ELEMENTS

OF

GEOMETRY,

TRANSLATED FROM THE FRENCH OF

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OF MILAN.

> Quærentem dictis quibus ......
> Clara tuæ poſſim præpandere lumina menti,
> Res quibus occultas penitus conviſere poſſis.
> LUCR.

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ADVERTISEMENT.

THE Tranſlator of this Work offers it to his Countrymen in their native tongue, becauſe it appears to him to be one of the moſt ſucceſsful attempts which has ever been made to facilitate the Study of Geometry. Originality of plan, clearneſs of arrangement, ſimplicity of demonſtration, and an elegant conciſeneſs of expreſſion, will, he apprehends, be found to be the leading characters of theſe Elements.

Thoſe who know where the difficulties of an undertaking of this kind principally lie, will eaſily pardon the liberties which the Author has taken, in making uſe of Numbers in the doctrine of Proportion, and, in employing the methods of Exhauſtions and Indiviſibles in demonſtrating the properties of Solids, when they perceive that by this means he has enabled the learner to make himſelf maſter of the principal elementary propoſitions, with the moſt perfect

perfect facility, and at the ſame time with a degree of accuracy ſufficient for every purpoſe of practice. Nor will it be thought too bold an innovation, that in a mathematical work the Author makes uſe of Phyſical Points, *and theſe too of different magnitudes; if it be obſerved, that he always expreſſes by this term the idea commonly affixed to* Aliquot Parts, *and that he only applies it to commenſurable lines or figures actually exiſting.*

It is not the Editor's idea, that this work can ſuperſede the uſe of the Elements of Euclid, which have a ſtrictneſs of proof, and univerſality of application, of which none of the more eaſy methods of demonſtration, invented by moderns can boaſt. But he is of opinion, that it may enable thoſe who are employed in the practical uſe of the Mathematical Arts, at an eaſy expence of time and attention, to acquire a general knowledge of the theory on which they are founded; and that it may be of great benefit to young Geometricians, in introducing them to a ready acquaintance with the terms and principles of this difficult ſcience, and preparing them for making further advances in mathematical knowledge.

THE AUTHOR's

PREFACE.

I HAVE neither vanity, nor humility enough to publiſh theſe Elements for the ſake of obtaining the name of Author: in this undertaking, utility is my ſole object. Nor indeed could any inferior motive have been ſufficient, to ſupport me under the tedious labours, to which my deſign has obliged me to ſubmit. What I here offer to the Public, is the reſult of numberleſs efforts, obſervations, and corrections, continued through twenty years. Though profeſſionally engaged in explaining the ſublimer parts of modern Geometry in one of the firſt Univerſities in Europe, I have not thought it beneath me to dedicate my leiſure hours to a work of this kind: for, the farther I have carried my reſearches, the more fully I have been convinced, of what all who are not wholly ſtrangers to the ſubject muſt perceive, that well-written Elements of Geometry were ſtill among the *deſiderata* of ſcience.

If theſe Elements ſhould fail of obtaining applauſe on the firſt curſory peruſal, I ſhall not be diſappointed. The ſuperficial reader will eaſily perſuade himſelf, that they are only a ſervile repetition of what was ſaid two thouſand years ago by the great Father of Geometry. Were this the caſe, I ſhould at leaſt have the merit of having convinced myſelf by long experience, of the preference due to the method of Euclid above any other which has been ſubſtituted in its room. But good judges, who know how to diſcriminate more accurately, will perhaps be of a different opinion. They will be aware of the rocks that were to be ſhunned, and the difficulties that were to be ſurmounted, in executing this deſign; and will eaſily perceive, that without indulging an affectation of novelty, I have often been under the neceſſity of ſtriking out into unbeaten paths; by which means obſtacles have ſometimes perhaps been avoided, which had before frequently embarraſſed both the Pupil and the Teacher.

All the Elementary Propoſitions, which can be of any uſe in a complete courſe of Mathematics, are here reduced to about an hundred. To make the proper ſelection required judgment, ſkill, and attention. How far I have ſucceeded in the attempt, muſt be left to the Public to determine; and the ſucceſs of my work will be the teſt of their approbation.

ELEMENTS

OF

GEOMETRY.

DEFINITIONS.

1. A *Solid* is that which hath length, breadth and thickneſs. A book, for inſtance, is a ſolid, ſince it is long, broad and thick.

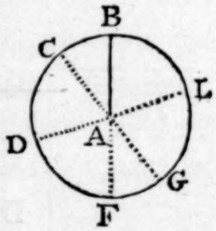
2. A *Surface* is that which has length and breadth, but no thickneſs. A leaf of fine paper may repreſent a ſurface.

3. A *Line* is that which has length, but neither breadth nor thickneſs. A hair may repreſent a line: a thread drawn ſtraight, a right line.

4. A *Point*

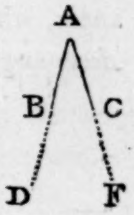
4. A *Point* has neither length, breadth, nor thickneſs. A fine grain of ſand may repreſent a point.

5. If a line AB is carried about a point A, in ſuch manner, that its other extremity paſſes from B to C, from C to D, &c. the point B, in its revolution, will deſcribe a curve line, BCDFGLB. This curve line is called the *Circumference* of a circle. The *Circle* is the ſpace incloſed by this circumference. The point A, which in the formation of the circle is at reſt, is called the *Center.* The right lines AC, AD, AF, &c. drawn from the center to the circumference, are called *Radii.* A *Diameter* is a right line which paſſes through the center, and is terminated both ways by the circumference: the line DAL, for example, is a diameter. An *Arc* is a part of the circumference, as FG.



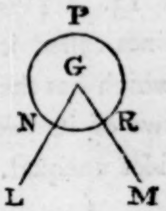
6. The circumference of a circle is divided into 360 equal parts, which are called *Degrees*; every degree is divided into 60 parts, which are called *Minutes*; every minute is divided into 60 parts, which are called *Seconds.* A large circumference has the ſame number of degrees with a ſmall one, but the degrees in the former are greater than thoſe in the latter.

7. Two right lines which are drawn from the ſame point, and recede from each other, form an opening which is called an *Angle*. An angle is commonly expreſſed by three letters; and it is uſual to place in the middle the letter which marks the vertical point of the angle; thus we ſay, the angle BAC, and not the angle ABC, or ACB.

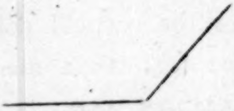


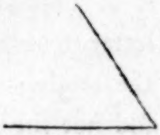
8. The magnitude of an angle does not depend upon the length of the lines which form it, but upon their diſtance from each other. How far ſoever the lines AB, AC are extended, the angle remains the ſame. One angle is greater than another, when the lines which form it are more diſtant. The angle BAL (ſee fig. def. 5.) is greater than the angle CAB, becauſe the lines AB, AL, are more diſtant from each other than the lines AC, AB. If the legs of a pair of compaſſes be a little ſeparated, an angle is formed; if they be opened wider, the angle becomes greater; if they be brought nearer, the angle becomes leſs.

9. If a point of the compaſſes be applied to the point G, and a circumference NRP be deſcribed, the arc NR, contained within the two lines GL, GM, will meaſure the magnitude of the angle LGM. If the arc NR, for example, is an arc of 40 degrees, the angle LGM is an angle of 40 degrees.

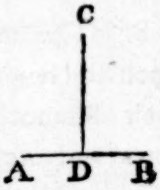


10. There are three kinds of angles; a *Right* angle, which is an angle of 90 degrees; an *Obtuſe* angle, which contains more than 90 degrees; and an *Acute* angle, which contains leſs than 90 degrees.

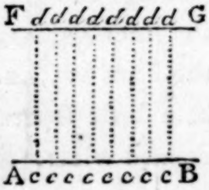




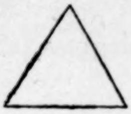
11. One line is *Perpendicular* to another line, when the two angles which it makes with that other line are equal: thus, the line CD is perpendicular to the line AB, if the angles CDA, CDB contain an equal number of degrees.

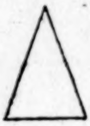


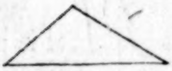
12. Two lines are *Parallel*, when all perpendiculars drawn from one to the other are equal: thus, the lines FG, AB are parallel, if all the perpendiculars, *c*, *d*, &c. are equal.



13. A *Triangle* is a ſurface encloſed by three right lines called its *Sides*. An *equilateral* triangle is that which has the three ſides equal: an *iſoſceles* triangle has two of its ſides equal: a *ſcalene* triangle has its three ſides unequal.

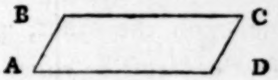




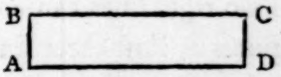


14. A *Quadrilateral* figure is a ſurface incloſed by four right lines, which are called its ſides.

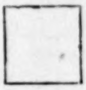
15. A *Parallelogram* is a quadrilateral figure, which has its oppoſite ſides parallel; thus, if the ſide BC is parallel to the ſide AD, and the ſide AB to the ſide DC, the quadrilateral figure ABCD is called a parallelogram.



16. A *Rectangle* is a quadrilateral figure, all the angles of which are right angles as ABCD.



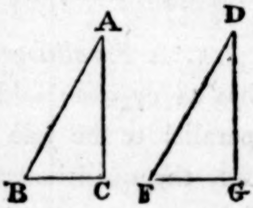
17. A *Square* is a quadrilateral figure, the ſides of which are all equal, and all its angles right angles.



18. A *Trapezium* is any quadrilateral figure not a parallelogram.

19. Thoſe figures are *Equal*, which incloſe an equal ſpace: thus, a circle and a triangle are equal, if the ſpace included within the circumference of the circle be equal to that contained in the triangle.

20. Thoſe figures are *Identical*, which are equal in all their parts, that is, which have their angles equal, and their ſides equal, and incloſe equal ſpaces, as BAC, FDG.



It is manifeſt that two figures are identical, which, being placed one upon the other, perfectly coincide; for, in that caſe, they muſt be equal in all their parts.

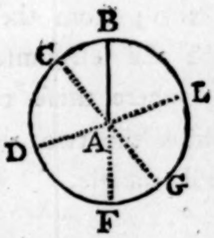
N. B. A line, uſed ſimply, always denotes a right line.

AXIOM. Two right lines cannot entirely incloſe a ſpace: this requires at leaſt three lines.

OF RIGHT LINES AND RECTILINEAL FIGURES.

PROPOSITION I.

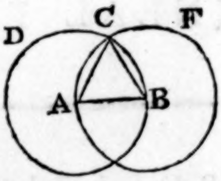
THE radii of the ſame circle are all equal.



The revolution of the line AB about the point A being neceſſary to form the circle BCDFGLB (def. 5.) when, in revolving, the point B is upon the point C, the whole line AB muſt be upon the line AC; otherwiſe two right lines would incloſe a ſpace, which is impoſſible: whence the radius AC is equal to the radius AB. In like manner it may be proved, that the radii AD, AF, AG, &c. are equal to AB: they are therefore equal among themſelves.

PROP. II.

To deſcribe an equilateral triangle upon a given line.



Let AB be the given line, upon which it is required to deſcribe a triangle, the three ſides of which are equal.

From the point A, with the radius AB, deſcribe the circumference BCD; from the point B, with the radius BA, deſcribe the circumference ACF; and from the point C, where theſe two circumferences cut each other, draw the two right lines CA, CB: ACB is an equilateral triangle,

For, the line AC is equal to the line AB,* [1] becauſe theſe two lines are radii of the ſame circle BCD: And, the line BC is equal to the line AB becauſe [1] theſe two lines are radii of the ſame circle ACF. Whence, the lines AC and BC, being each equal to the ſame line AB, are equal to one another; and all the three ſides of the triangle ACB are equal; that is, the triangle is equilateral.

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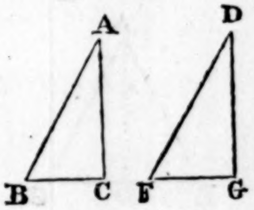
* The figures refer to the preceding propoſitions.

PROP. III.

Triangles which have two ſides and the angle contained by them equal, are identical.

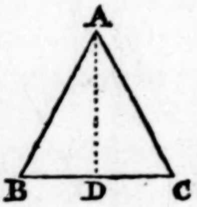


In the two triangles BAC, FDG, if the ſide DF be equal to the ſide AB, and the ſide DG equal to the ſide AC, and alſo the angle D equal to the angle A, the two triangles are identical.

Suppoſe the triangle FDG placed upon the triangle BAC, in ſuch manner that the ſide DF falls exactly upon the ſide equal to it, AB. Since the angle D is equal to the angle A, the ſide DG muſt fall upon the ſide equal to it, AC: alſo, the point F will be upon the point B, and the point G upon the point C: conſequently, the line FG muſt fall wholly upon the line BC; otherwiſe two right lines would incloſe a ſpace, which is impoſſible. The three ſides of the triangle FDG, therefore, coincide in all points, with the three ſides of the triangle BAC; and the two triangles have their ſides and angles equal, and incloſe an equal ſpace, that is, (def. 20.) they are identical.

PROP. IV.

In an iſoſceles triangle the angles at the baſe are equal.



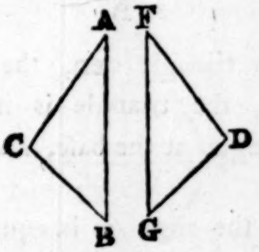
Let the triangle BAC have its ſides AB, AC, equal; the angles B and C, at the baſe, are alſo equal.

Conceive the angle A to be biſected by the right line AD.

In the triangles BAD, DAC, the ſides AB, AC are, by ſuppoſition, equal; the ſide AD is common to the two triangles; and the angles at A are ſuppoſed equal. Theſe two triangles have then the two ſides, and the angle contained by them, equal: they are therefore 3 identical, or have all their parts equal: whence the angles B and C are equal.

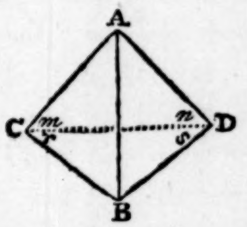
PROP. V.

Triangles which have their three ſides equal, are identical.



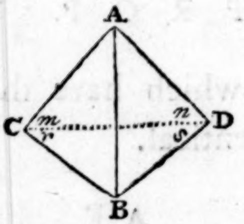
In the two triangles ACB, FDG, let the ſide AC be equal to the ſide FD, the ſide CB equal to the ſide DG, and the ſide AB to the ſide FG; theſe two triangles are identical.

Let the two triangles be ſo joined, that the ſide FG ſhall coincide with the ſide AB, and draw the right line CD.



Since, in the triangle CAD, the ſide AC is equal to the ſide AD, the triangle is iſoſceles; whence (def. 13.) the angles at the baſe, *m* and *n*, are equal.

Since,



Since, in the triangle CBD, the ſide BC is equal to the ſide BD, the triangle is iſoſceles; whence (def. 13.) the angles at the baſe, *r* and *s*, are equal.

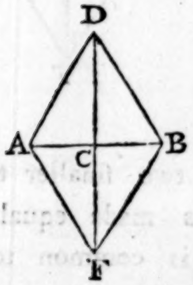
And becauſe the angle *m* is equal to the angle *n*, and the angle *r* equal to the angle *s*, the whole angle C is equal to the whole angle D.

Laſtly, becauſe in the two triangles ACB, ADB, the ſide AC is equal to the ſide AD, and the ſide CB equal to the ſide DB, and alſo the angle C equal to the angle D, theſe two triangles have two ſides and the contained angle equal, and are therefore 3 identical.

PROP.

PROP. VI.

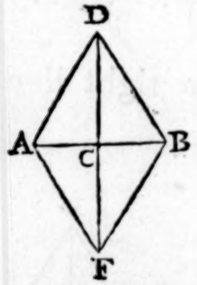
To divide a right line into two equal parts.



Let the right line, which it is required to divide into two equal parts, be AB. Upon AB draw [2] the equilateral triangle ADB; and on the other ſide of the ſame line AB draw the equilateral triangle AFB; draw alſo the right line DF; AC is equal to CB.

In the two larger triangles DAF, DBF, the ſides DA, DB, are equal, becauſe they are ſides of an equilateral triangle; the ſides AF, BF, are equal for the ſame reaſon; and the ſide DF is common to the two triangles. Theſe two triangles have then their three ſides equal, and conſequently [5] are identical, or have all their parts equal; whence the two angles at D are equal.

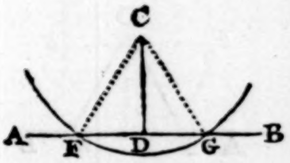
Again,



Again, in the two ſmaller triangles ADC, CDB, the ſide DA is made equal to the ſide DB; and the ſide DC is common to the two triangles; alſo, the two angles at D are equal. Theſe two triangles have then two ſides, and the contained angle, equal; they are therefore 3 identical, and AC is equal to CB, that is, AB is biſected.

PROP. VII.

From a given point, out of a right line, to draw a perpendicular to that line.



Let c be the point from which it is required to draw a perpendicular to the right line AB.

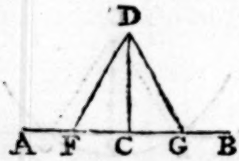
From the point c deſcribe an arc of a circle which ſhall cut the line AB in two points, F and G. Then biſect the line FG, and to D, the point of diviſion, draw the line CD: this line is perpendicular to the line AB. Draw the lines CF, CG.

In the triangles FCD, DCG, the ſides, CF, CG are equal, [1] becauſe they are radii of the ſame circle; the ſides FD, DG are equal, becauſe FG is biſected; and the ſide CD is common. Theſe two triangles then, having the three ſides equal, are 5 identical. Whence (def. 20.) the angle CDA is equal to the angle CDB, and conſequently (def. 11.) the line CD is perpendicular to the line AB.

PROP.

PROP. VIII.

From a given point in a right line, to raiſe a perpendicular upon that line.



From the point C let it be required to raiſe a perpendicular upon the right line AB.

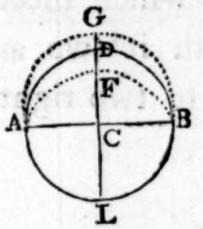
In AB take, at pleaſure, CF equal to CG; upon the line FG deſcribe an equilateral triangle FDG, and draw the line CD; this line will be perpendicular to AB.

In the triangles FDC, CDG, the ſides DF, DG are equal, becauſe they are ſides of an equilateral triangle; the ſides FC, CG are equal by conſtruction; and the ſide DC is common. Theſe two triangles, then, having the three ſides equal, are 5 identical. Therefore (def. 20.) the angle DCA is equal to the angle DCB, and conſequently (def. 11.) the line CD is perpendicular to the line AB.

PROP.

PROP. IX.

The diameter of a circle divides the circumference into two equal parts.



Let ADBLA be a circle; the diameter ACB biſects the circumference, that is, the arc ALB is equal to the arc ADB.

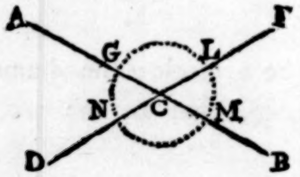
Conceive the circle to be divided, and the lower ſegment ACBLA to be placed upon the upper ACBDA; all the points of the arc ALB will fall exactly upon the arc ADB, and conſequently theſe two arcs will be equal.

For, if the point L, for inſtance, does not fall upon the arc ADB, it muſt fall either above this arc, as at G, or below it, as at F. If it falls on G, the radius CL will be greater than the radius CD; if it falls on F, the radius CL will be leſs than the radius CD; which [1] is impoſſible. The point L muſt then fall upon the arc ADB. In like manner, it may be proved, that all the other points of the arc ALB muſt fall upon the arc ADB: theſe two arcs are therefore equal.

PROP.

PROP. X.

A right line which meets another right line, forms with it two angles which are together equal to two right angles.



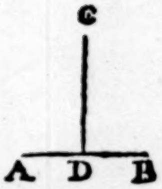
The line AC meeting the line DF, and forming with it the two angles ACD, ACF, theſe two angles are together equal to two right angles.

From the point C as a center deſcribe, at pleaſure, a circumference NGLMN.

The line NCL, being a diameter, divides the circumference 9 into two equal parts. The arc NGL is therefore half the circumference, which contains (def. 6.) 180, or twice 90, degrees. Therefore the angles ACD, ACF, which taken together are meaſured by the arc NGL, are twice 90 degrees, that is (def. 10.) are equal to two right angles.

PROP. XI.

A right line drawn perpendicularly to another right line, makes right angles with it.

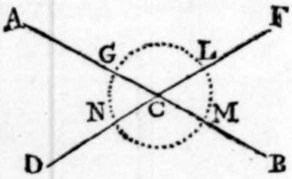


If the line CD be perpendicular to the line AB, the angle CDA is a right angle, and alſo the angle CDB.

For the line CD, meeting the line AB, forms with it two angles which are together [10] equal to two right angles; and theſe two angles are equal, becauſe CD is perpendicular to AB: wherefore each angle is a right angle.

PROP. XII.

If two lines cut each other, the vertical or oppoſite angles are equal.



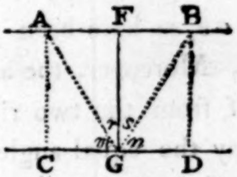
Let the lines AB, DF, cut each other at the point C; the angles ACD, FCB, which are called vertical or oppoſite angles, are equal.

From the point C as a center deſcribe, at pleaſure, a circumference NGLMN.

Since the line NCL is a diameter, the arc NGL is 9 half the circumference: alſo ſince GCM is a diameter, the arc GLM is 9 half the circumference: therefore the arcs NGL, GLM are equal. From theſe two arcs take away the common part GL, there will remain the arc NG equal to the arc LM: conſequently, the angles ACD, FCB, which are meaſured by theſe two arcs, are alſo equal.

P R O P. XIII.

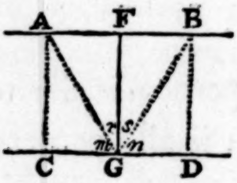
If a line is perpendicular to one of two parallel lines, it is alſo perpendicular to the other.



Let AB, CD be two parallel lines: if the line FG makes right angles with CD, it will alſo make right angles with AB.

Take at pleaſure GC equal to GD; at the points C and D, raiſe the perpendiculars, CA, DB; and draw the lines GA, GB.

In the two triangles ACG, BDG, becauſe the line AB is parallel to the line CD, the perpendiculars CA, DB are neceſſarily equal, as appears from the definition of parallel lines (def. 12.): the lines CG, DG, are equal by conſtruction; and the angles C and D are right angles. The two triangles ACG, BDG have then two ſides, and the contained angle, equal: they are therefore [3] identical. Whence the ſide GA is equal to the ſide GB, and the angle *m* equal to the angle *n*.

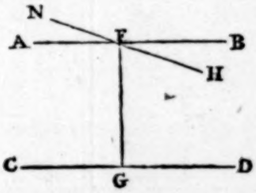


Again, in the triangles AGF, FGB, the ſide GA is equal to the ſide GB, as hath been proved; and the ſide GF is common. Moreover, the angle *r* is equal to the angle *s*: for, if from the two right angles FGC, FGD, be taken away the equal angles *m* and *n*, there will remain the equal angles *r* and *s*. The triangles AGF, FGB have then two ſides, and the contained angle, equal: they are therefore [3] identical. Wherefore the angles GFA, GFB are equal, and conſequently are right angles.

PROP.

PROP. XIV.

If one line is perpendicular to two other lines, theſe two lines are parallel.

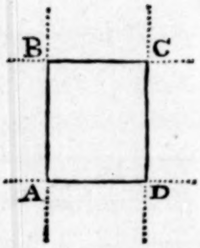


Let the line FG make right angles with the lines AB and CD; theſe two lines are parallel.

If the line AB be not parallel to the line CD, another line, as NH, may be drawn through the point F parallel to the line CD. But this is impoſſible; for if the line NH were parallel to the line CD, the line FG, making right angles with CD, would alſo [13] make right angles with NH; which cannot be, becauſe, by ſuppoſition, it makes right angles with AB.

PROP. XV.

The oppoſite ſides of a rectangle are parallel.



In the rectangle ABCD, the ſide BC is parallel to the ſide AD, and the ſide AB parallel to the ſide DC.

Produce each of the ſides both ways.

The line AB is perpendicular to the two lines BC, AD; the two lines BC, AD are therefore [14] parallel. In like manner, the line AD is perpendicular to the two lines AB, DC; the two lines AB, DC, are therefore [14] parallel.

PROP. XVI.

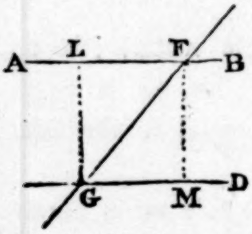
The oppoſite ſides of a rectangle are equal.

In the rectangle ABCD, [ſee fig. to the preceding propoſition] the ſide AB is equal to the ſide DC, and the ſide BC equal to the ſide AD.

For, ſince the ſide BC is parallel to the ſide AD, the perpendiculars AB, DC are (def. 12.) equal; and, ſince the ſide AB is parallel to the ſide DC, the perpendiculars BC, AD are equal.

PROP. XVII.

A right line, falling upon parallel lines, makes the alternate angles equal.



Let the line FG cut the parallels AB, GD; the angles AFG, FGD, which are called *alternate angles*, are equal.

From the point G draw GL perpendicular to the line AB; and from the point F draw FM perpendicular to the line GD.

Since the line GL is perpendicular to AB, it is alſo [13] perpendicular to the parallel line GD. In like manner, ſince the line FM is perpendicular to the line GD, it is alſo [13] perpendicular to the parallel line AB. Whence the quadrilateral figure GLFM is a rectangle, its four angles being right angles.

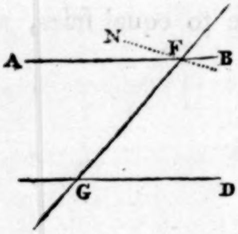
In the triangles GLF, FMG, the ſides LF, GM are equal, becauſe they are oppoſite ſides of the ſame rectangle;

rectangle: the ſides LG, FM, are equal for the ſame reaſon; and the ſide FG is common. The two triangles GLF, FMG, have then the three ſides equal, and conſequently 5 are identical. Wherefore the angle LFG, oppoſite to the ſide LG, is equal to the angle FGM, oppoſite to the ſide FM.

Remark. In identical triangles, the equal angles are always oppoſite to equal ſides, as appears in this propoſition.

PROP. XVIII.

If one right line, falling upon two others, makes the alternate angles equal, theſe two lines are parallel.

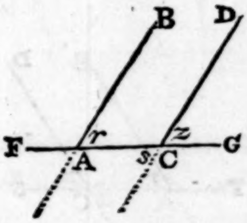


Let the alternate angles AFG, FGD, be equal, the lines AB, GD are parallel.

If the line AB is not parallel to the line GD, another line, as NF, may be drawn through the point F parallel to GD. But this is impoſſible: for, if the line NF were parallel to the line GD, the angle FGD would be [17] equal to the angle NFG, ſince theſe two angles would be alternate angles between two parallel lines; which cannot be, becauſe, by ſuppoſition, the angle FGD is equal to the angle AFG.

PROP. XIX.

If one right line falls upon two parallel right lines, it makes the interior angle equal to the exterior.



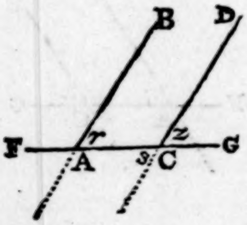
Let the line FG meet the parallel lines BA, DC, the interior angle *r* is equal to the exterior angle *z*.

Produce the lines BA, DC.

The angle *r* is [17] equal to the angle *s*, becauſe theſe are alternate angles, made by a right line falling upon two parallel lines: and the angles *s* and *z* are [12] equal, becauſe they are vertical or oppoſite angles: therefore the angle *r* is equal to the angle *z*.

PROP. XX.

If one right line, falling upon two other right lines, makes the internal angle equal to the external, theſe two lines are parallel.

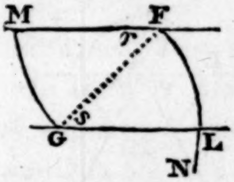


Let the internal angle *r* be equal to the external angle *z*, the lines BA, DC are parallel.

The angle *r* is equal to the angle *z* by ſuppoſition; and the angle *z* is [12] equal to the angle *s*, becauſe they are oppoſite angles. The alternate angles *r*, *s*, are therefore equal, and conſequently [18] the lines BA, DC are parallel.

PROP. XXI.

Through a given point to draw a line parallel to a given line.



Let G be the point through which it is required to draw a line parallel to the given line MF.

From any point G deſcribe, at pleaſure, the arc FN: from the point F, in which the arc FN cuts the line MF, with the diſtance GF, deſcribe the arc GM meeting the line MF in M: then make FL equal to GM, and draw the line GL; this line is parallel to the line MF.

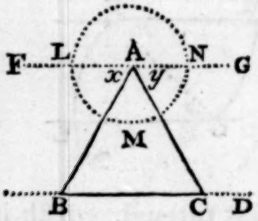
Draw the line GF.

The arcs GM, FL are equal by conſtruction; therefore the alternate angles *r*, *s*, which are meaſured by theſe arcs (def. 9.) are equal: and conſequently[18] the lines GL MF are parallel.

PROP.

PROP. XXII.

The three angles of a triangle are equal to two right angles.



In the triangle BAC, the three angles B, A, C, are together equal to two right angles.

Produce the ſide BC both ways; through the point A, draw a line FG parallel to BC; and from the point A, as a center, deſcribe any circumference LMN.

The angle B is [17] equal to the angle *x*, becauſe theſe are alternate angles made by a right line falling upon two parallel lines. For the ſame reaſon the angle C is equal to the angle *y*.

Becauſe LAN is a diameter, the arc LMN is half the circumference: therefore the three angles *x*, A, *y*, which are meaſured by this arc, are together equal to two right angles.

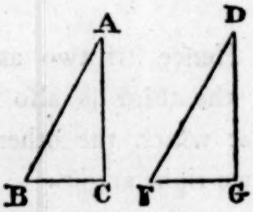
But the angle *x* is equal to the alternate angle B, and the angle *y* to the alternate angle C.

Therefore, ſubſtituting B for *x*, and C for *y*, the three angles B, A, C, are together equal to two right angles.

COROLLARY. Hence, if two angles of any triangle be known, the third is alſo found; ſince the third angle is that which the other two, taken together, want of two right angles.

PROP. XXIII.

If two triangles have two angles equal, they have alſo the third angle equal.



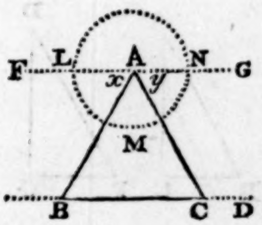
In the two triangles BAC, FDG, if the angle B is equal to the angle F, and the angle A equal to the angle D, the angle C will alſo be equal to the angle G.

Since the angle C (cor. 22.) is that which the angles B and A, together, want of two right angles; and ſince the angle G is that which F and D, together, want of two right angles; the angles B and A being equal to the angles F and D, the angle C muſt be equal to the angle G.

PROP.

PROP. XXIV.

The exterior angle of any triangle is equal to the two interior and oppoſite angles taken together.



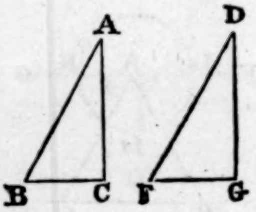
In the triangle BAC, produce one of the ſides, BC, the angle ACD, which is called *exterior*, is equal to the two *interior and oppoſite* angles B and A, taken together.

The line AC meeting the line BD, forms with it two angles, which are together [10] equal to two right angles: the angle ACB is therefore that which the angle ACD wants of two right angles. But the angle ACB is (cor 22.) alſo that which the two angles B and A, together, want of two right angles. Wherefore the angle ACD is equal to the two angles B and A taken together.

PROP.

PROP. XXV.

Triangles which have two angles, and the ſide which lies between them, equal, are identical.



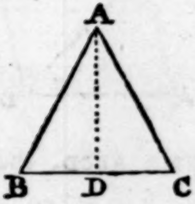
In the two triangles BAC, FDG, if the angle F is equal to the angle B, the angle G equal to the angle C, and the ſide FG equal to the ſide BC, theſe two triangles are identical.

Conceive the triangle FDG placed upon the triangle BAC, in ſuch manner that the ſide FG ſhall fall exactly upon the equal ſide BC. Since the angle F is equal to the angle B, the ſide FD muſt fall upon the ſide BA: and ſince the angle G is equal to the angle C, the ſide GD muſt fall upon the ſide CA. Thus the three ſides of the triangle FDG will be exactly placed upon the three ſides of the triangle BAC; and conſequently the two triangles 5 are identical.

PROP.

PROP. XXVI.

If two angles of a triangle are equal, the ſides oppoſite to theſe angles are alſo equal.

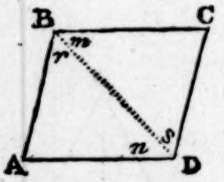


Conceive the angle A to be biſected by the line AD.

In the triangles BAD, DAC, the angle B is equal to the angle C, by ſuppoſition; and the angles at A are alſo equal. Theſe two triangles have then two angles equal; the third angle will therefore 23 be equal: whence the angles at D are equal. Moreover, the ſide AD is common to the two triangles. Theſe two triangles therefore, having two angles, and the ſide which lies between them equal, are 25 identical. Whence the ſide AB is equal to the ſide AC.

PROP. XXVII.

The oppoſite ſides of a parallelogram are equal.



In the parallelogram ABCD, the ſide AB is equal to the ſide DC, and the ſide BC equal to the ſide AD.

Draw the line BD which is called the *Diagonal.*

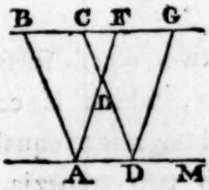
Becauſe BC is parallel to AD, the alternate angles *m* and *n* are equal. In like manner, becauſe AB is parallel to DC, the alternate angles *r* and *s* are equal. Alſo, the ſide BD is common to the two triangles BAD, BCD. Theſe two triangles have then two angles, and the ſide which lies between them, equal, and are therefore [3] identical. Wherefore the ſide AB, oppoſite to the angle *n*, is [26] equal to the ſide DC, oppoſite to the equal angle *m*; and the ſide BC, oppoſite to the angle *s*, is equal to the ſide AD, oppoſite to the equal angle *r*.

COR. Hence it follows, that the diagonal biſects the parallelogram: for, the triangles BAD, BCD, having the three ſides equal, are identical.

PROP.

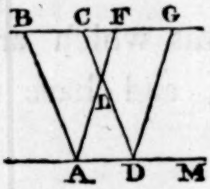
PROP. XXVIII.

Parallelograms which are between the ſame parallels, and have the ſame baſe, are equal.



Let the two parallelograms ABCD, AFGD, be between the ſame parallels BG, AM, and upon the ſame baſe AD; the ſpace encloſed within the parallelogram ABCD, is equal to the ſpace encloſed within the parallelogram AFGD.

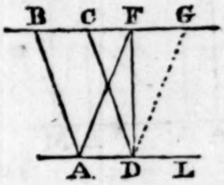
In the two triangles BAF, CDG, the ſide BA of the former triangle is equal to the ſide CD of the latter, becauſe they are oppoſite ſides of the ſame parallelogram. For the ſame reaſon, the ſide FA is equal to the ſide GD. Moreover, BC is equal to AD, becauſe they are oppoſite ſides of the ſame parallelogram. For the ſame reaſon, AD is equal to FG. BC is therefore equal to FG. If to both theſe CF be added, BF will be equal to CG. Whence the two triangles BAF, CDG, having the three ſides equal 5 are identical, and conſequently have equal ſurfaces.



If from theſe two equal ſurfaces be taken the ſmall triangle CLF, which is common, there will remain the trapezium ABCL equal to the trapezium LFGD. To theſe two trapezia, add the triangle ALD, and the parallelogram ABCD will be equal to the parallelogram AFGD.

PROP. XXIX.

If a triangle and a parallelogram are upon the ſame baſe, and between the ſame parallels, the triangle is equal to half the parallelogram.



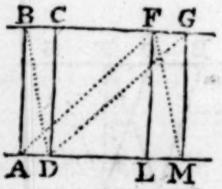
Let the parallelogram ABCD, and the triangle AFD, be upon the ſame baſe AD, and between the ſame parallels BG, AL; the triangle AFD is half the parallelogram ABCD.

Draw DG parallel to AF.

Becauſe the parallelogram AFGD is biſected by the diagonal FD, (prop. 27. cor.) the triangle AFD is half the parallelogram AFGD. But the parallelogram AFGD is equal to the parallelogram ABCD, becauſe theſe two parallelograms are upon the ſame baſe and between the ſame parallels: therefore the triangle AFD is equal to half the parallelogram ABCD.

PROP. XXX.

Parallelograms which are between the ſame parallels, and have equal baſes, are equal.



Let the two parallelograms ABCD, LFGM be between the ſame parallels BG, AM, and have the equal baſes AD, LM; theſe two parallelograms are equal.

Draw the lines AF, DG.

Becauſe AD is equal to LM, and LM to FG, AD is equal to FG; and they are parallel by conſtruction: alſo AF and DG are parallel; for, if DG be not parallel to AF, another line may be drawn parallel to it; whence FG will become greater or leſs than AD; which is impoſſible, becauſe FG has been proved to be equal to AD. AF and AG are therefore parallel, and AFGD a parallelogram.

Now the parallelogram ABCD is [28] equal to the parallelogram AFGD, becauſe theſe two parallelograms are between the ſame parallels, and have the ſame baſe AD. And the parallelogram AFGD is equal to the parallelogram LFGM, becauſe theſe two parallelograms are between the ſame parallels, and have the ſame baſe FG. The parallelogram ABCD is therefore equal to the parallelogram LFGM.

PROP. XXXI.

Triangles which are between the ſame parallels and have equal baſes, are equal.

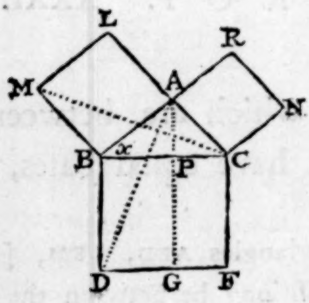
Let the two triangles ABD, LFM, [ſee fig. to the preceding propoſition] be between the ſame parallels BG, AM, and upon the equal baſes AD, LM, theſe two triangles are equal.

Draw DC parallel to AB, and MG parallel to LF.

The two parallelograms ABCD, LFGM, are equal, [30] becauſe they are between the ſame parallels, and have equal baſes. But the triangle ABD is [29] one half of the parallelogram ABCD; and the triangle LFM is one half of the parallelogram LFGM. Theſe two triangles are therefore equal.

PROP. XXXII.

In a right-angled triangle the ſquare of the hypothenuſe, or ſide ſubtending the right angle, is equal to the ſquares of the ſides which contain the right angle.



In the triangle BAC, let the angle A be a right angle. Upon the hypothenuſe BC deſcribe the ſquare BDFC; upon the ſide AB deſcribe the ſquare ALMB, and upon the ſide AC, the ſquare ARNC; the ſquare BDFC is equal to the two ſquares ALMB, ARNC taken together.

Draw the right lines MC, AD; and draw AG parallel to BD.

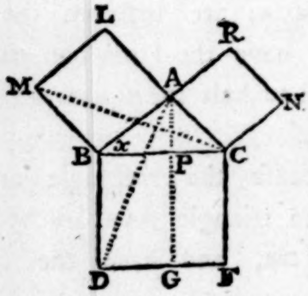
Becauſe the ſquare or parallelogram MLAB, and the triangle MCB, are between the ſame parallels LC, MB, and have the ſame baſe MB, the triangle MCB is [29] equal to half the ſquare ALMB.

Again, becauſe the rectangle or parallelogram DGPB, and the triangle DAB are between the ſame parallels GA, DB, and have the ſame baſe DB, the triangle DAB is [29] equal to half the rectangle DGBP.

Further, ſince the ſide MB of the triangle MBC and the ſide AB of the triangle ABD, are ſides of the ſame ſquare, they are (def. 17.) equal. Alſo, ſince the ſide BC of the firſt triangle, and the ſide BD of the ſecond triangle are ſides of the ſame ſquare, they are equal. And becauſe the angle MBC of the firſt triangle is compoſed of a right angle and the angle x, and the angle ABD of the ſecond triangle is compoſed of a right angle, and the ſame angle x, therefore theſe two angles, contained between the equal ſides, MB, BC, and AB, BD, are equal. Wherefore, the two triangles MBC, ABD, having two ſides and the contained angle equal, are 3 identical, and conſequently equal.

But the triangle MBC is half the ſquare MLAB; and the triangle ABD is half the rectangle BDGP. The ſquare and the rectangle are therefore equal.

In



In the ſame manner it may be demonſtrated, that the ſquare ARNC and the rectangle CFGP are equal: whence it follows, that the whole ſquare BDFC is equal to the two ſquares MLAB, ARNC, taken together.

OF

CIRCLES.

DEFINITIONS.

1. A RIGHT line (ſee fig. prop. XXXIII. AB) terminated both ways by the circumference of a circle, is called a *Chord*.

2. A line (ſee fig. prop. XXXIX. AB) which meets the circumference in one point only, is called a *Tangent*; and the point T is called the *Point of Contact*.

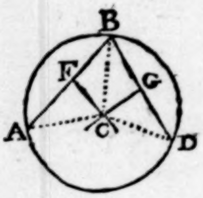
3. An angle (ſee fig. prop. XXXIII. ABD) which has its vertex in the circumference of a circle, is called *an Angle in the Circle*.

4. A part of a circle confined between two radii, (ſee fig. prop. XXXIV. ACBFA) is called a *Sector*.

5. A part of a circle (ſee fig. prop. XXXV. AGBDA) terminated by a chord, is called a *Segment of a Circle*.

PROP. XXXIII.

To draw the circumference of a circle through three given points.



Let there be three given points, A, B, D, through which it is required to draw the circumference of a circle.

Draw the right lines AB, BD, and biſect them: from the points of diviſion, F, G, raiſe the perpendiculars FC, GC; and at the point C, with the radius CA, deſcribe the circumference of a circle: this circumference will paſs through the points B and D. Draw the lines CA, CB, CD.

In the triangles CFA, CFB, the ſide FA is equal to the ſide FB by conſtruction; the ſide FC is common; and the two angles at F are right angles. Theſe two triangles have then two ſides, and the angle contained by them, equal: they are therefore 3 identical. Conſequently the ſide CB is equal to the ſide CA.

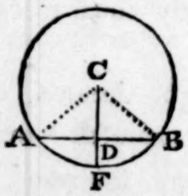
For the ſame reaſon, the triangles CGB, CGD are alſo identical. whence the ſide CD is equal to the ſide CB, and conſequently equal to CA.

And ſince the right lines CB, CD are equal to the right line CA, it is manifeſt 1 that the circumference which paſſes through the point A, muſt alſo paſs through the point D.

PROP.

PROP. XXXIV.

If a radius biſects a chord, it is perpendicular to that chord.



If the radius CF biſects the chord AB, the angles CDA, CDB are right angles. Draw the radii CA, CB.

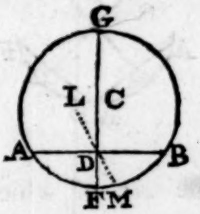
In the triangles CDA, CDB, the ſides CA, CB, being radii, are [1] equal; the ſides AD, DB, are equal by ſuppoſition; and the ſide CD is common. Theſe two triangles, having the three ſides equal, are therefore [5] identical. Whence the angles CDA, CDB, are equal, and conſequently [10] are right angles.

COR. The two angles at C are alſo [5] equal.

Hence it appears, that any angle ACB may be biſected, by deſcribing from its vertex C as the center, with any radius AC, an arc AFB; biſecting the chord of that arc AB; and then drawing, from the point of diviſion D, the right line CD: for it may then be ſhewn as in the propoſition, that the triangles ACD, DCB are identical, and conſequently the angles at C equal.

PROP. XXXV.

To find the center of a circle.

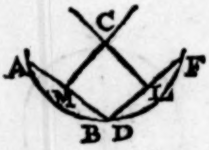


Let the circle of which it is required to find the center be AGBF. Draw any chord AB; biſect it, and from the point of diviſion D, raiſe a perpendicular FG: this line will paſs through the center, and conſequently if it be biſected, the point of diviſion will be the center.

If the center of the circle be not in the line FG, it muſt be ſomewhere out of it, for inſtance at the point L. But this is impoſſible; for if the point L were the center, the right line LM would be a radius; and ſince this line biſects the chord AB, it is 34 perpendicular to AB; which cannot be, ſince CD is perpendicular to AB.

PROP. XXXVI.

To find the center of an arc of a circle.

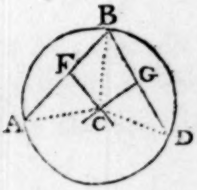


Let ABDF be the arc of which it is required to find the center. Draw any two chords, AB, DF; biſect them, and from the points of diviſion, raiſe the perpendiculars MC, LC; the point C, in which theſe two perpendiculars cut each other, is the center of the arc.

For 35 the perpendicular MC, and the perpendicular LC, both paſs through the center of the ſame circle; this center muſt therefore be the point C, which is the only point common to the two perpendiculars.

PROP. XXXVII.

If three equal lines meet in the ſame point within a circle, and are terminated by its circumference, they are radii of that circle.

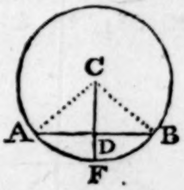


The lines CA, CB, CD, drawn from the ſame point C, within a circle, and terminated by it, being equal, the point C is the center of the circle. Draw the lines AB, BD; biſect them, and let the points of diviſion be F, G; and draw the lines CF, CG.

In the triangles CFA, CFB, the ſides CA, CB are equal by ſuppoſition; the ſides FA, FB are equal by conſtruction; and the ſide CF is common. Theſe two triangles have then the three ſides equal: they are therefore 5 identical. Whence the two angles at F are equal, and the line FC (def. 11.) is perpendicular to the chord AB. And ſince this perpendicular biſects the chord AB, it muſt 35 paſs through the center of the circle. In like manner it may be demonſtrated, that the line GC alſo paſſes through the center. Whence the point C is the center of the circle, and CA, CB, CD are radii.

PROP. XXXVIII.

If the radius of a circle is perpendicular to a chord, the radius biſects both the chord and the arc of the chord.



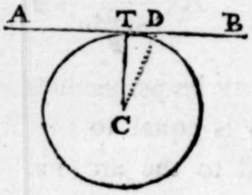
Let the radius CF be perpendicular to the chord AB; the right line AD is equal to the right line DB, and the arc AF equal to the arc FB. Draw the right lines CA, CB.

In the large triangle ACB, the ſide CA is [1] equal to the ſide CB, becauſe they are radii of the ſame circle: the angle A is therefore equal to the angle B.

In the two ſmaller triangles CDA, CDB, the angle A is [4] equal to the angle B; the angles at D are right angles, and therefore equal; and the angles at C are conſequently [23] equal. Alſo, the ſide CA is equal to the ſide CB, and the ſide CD is common. Theſe two triangles having then two ſides, and the angle contained by them, equal, are [3] identical; whence the ſide AD is equal to the ſide DB. Again, ſince the angles ACF, BCF are equal, the arcs AF, BF, which meaſure theſe angles, are alſo equal. The chord AB, and the arc AFB, are therefore biſected by the radius CF.

PROP. XXXIX.

A right line perpendicular to the extremity of a radius, is a tangent to the circle.



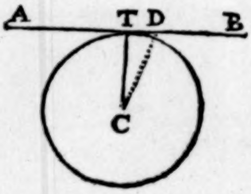
Let the line AB paſs through the extremity of the radius CT, in ſuch manner that the angles CTA, CTB ſhall be right angles: this line AB touches the circumference only in one point T. If AB touches the circumference in any other point, let it be D, and draw the line CD.

In the right-angled triangle CTD, the ſquare of the hypothenuſe CD is equal to the two ſquares of CT and TD taken together. The ſquare of CD is therefore greater than the ſquare of CT, and conſequently the line CD is greater than the line CT, which is a radius. Therefore the point D is out of the circumference. And in like manner it may be ſhewn, that every point in the line AB is out of the circumference except T; AB is therefore a tangent to the circle.

COR. Hence it follows, that a perpendicular is the ſhorteſt line which can be drawn from any point to a given line; ſince the perpendicular CT is ſhorter than any other line which can be drawn from the point C to the line AB.

PROP. XL.

If a right line be drawn touching a circumference, a radius drawn to the point of contact will be perpendicular to the tangent.

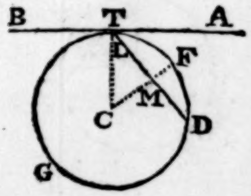


Let the line AB touch the circumference of a circle in a point T; the radius CT is perpendicular to the tangent AB.

For, all other lines drawn from the point C to the line AB, muſt paſs out of the circle to arrive at this line. The line CT is therefore the ſhorteſt which can be drawn from the point C to the line AB, and conſequently (39 Cor.) is perpendicular to the line AB.

PROP. XLI.

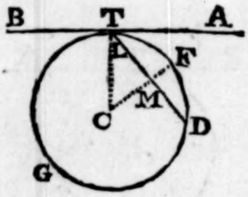
The angle formed by a tangent and chord, is meaſured by half the arc of that chord.



Let BTA be a tangent, and TD a chord drawn from the point of contact T; the angle ATD is meaſured by half the arc TFD; and the angle BTD is meaſured by half the arc TGD. Draw the radius CT to the point of contact, and the radius CF perpendicular to the chord TD.

The radius CF, being perpendicular to the chord TD, [38] biſects the arc TFD. TF is therefore half the arc TFD.

In the triangle CML, the angle M being a right angle, the two remaining angles are [22] equal to a right angle; whence the angle C is that which the angle L wants of a right angle. On the other ſide, ſince the radius CT is perpendicular to the tangent BA, the angle ATD is alſo that which the angle L wants of a right angle. The angle ATD is therefore equal to the angle C. But the angle C is meaſured by the arc TF; conſequently, the angle ATD is alſo meaſured by the arc TF, which is half of TFD, the arc of the chord TD.

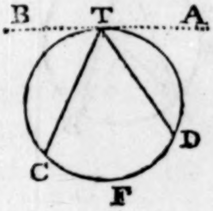


Again, the line TD forms with the line BA two angles ATD, BTD, which are [10] equal to two right angles, and conſequently are meaſured by half the circumference. But the angle ATD is meaſured by half the arc TFD. The angle BTD muſt therefore be meaſured by half the arc TGD, ſince theſe two halves of arcs make up half the circumference.

PROP.

PROP. XLII.

An angle at the circumference of a circle, is meaſured by half the arc by which it is ſubtended.



Let CTD be the angle at the circumference; it has for its meaſure, half the arc CFD by which it is ſubtended.

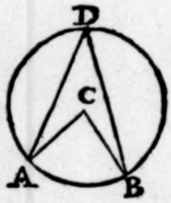
Suppoſe a tangent paſſing through the point T.

The three angles at T are meaſured by half the circumference (ſee prop. 22.) but the angle ATD is meaſured 41 by half the arc TD, and the angle BTC by half the arc TC; conſequently the angle CTD muſt be meaſured by half the arc CFD, ſince theſe three halves of arcs make up half the circumference.

PROP.

PROP. XLIII.

The angle at the center of a circle is double of the angle at the circumference.

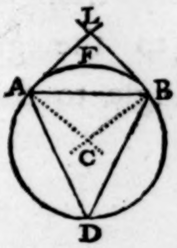


Let the angle at the circumference ADB, and the angle at the center ACB, be both ſubtended by the ſame arc AB; the angle ACB is double of the angle ADB.

For, the angle ACB is meaſured by the arc AB; and the angle ADB is [42] meaſured by half the ſame arc AB: the angle ACB is therefore double of the angle ADB.

PROP. XLIV.

Upon a given line, to deſcribe a ſegment of a circle containing a given angle.



Let AB be the given line, and G the given angle: it is required to draw ſuch a circumference of a circle through the points A and B, that the angle D ſhall be equal to the angle G.

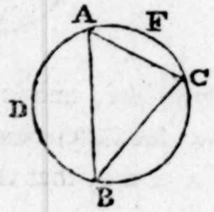
For this purpoſe, draw the lines AL, BL, in ſuch manner that the angles A and B ſhall be equal to the angle G; at the extremities of LA, LB, raiſe the perpendiculars AC, BC; and from the point C in which theſe two perpendiculars cut each other, with the radius CA or CB, deſcribe the circumference ADB; the angle D will be equal to the angle G.

The angle LAB, formed by the tangent AL and the chord AB is 41 meaſured by the half of the arc AFB; and the angle D at the circumference is alſo meaſured 43 by the half of the arc AFB: the angle D is therefore equal to the angle LAB. But the angle LAB is made equal to the angle G; the angle D is therefore equal to the angle G.

PROP.

PROP. XLV.

In every triangle, the greater ſide is oppoſite to the greater angle, and the greater angle to the greater ſide.



In the triangle ABC, if the ſide AB is greater than the ſide AC, the angle C oppoſite to the ſide AB, will be greater than the angle B oppoſite to the ſide AC. Draw the circumference of a circle through the three points A, C, B.

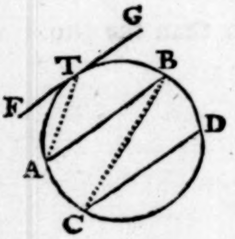
Since the chord AB is greater than the chord AC, it is manifeſt that the arc ADB is greater than the arc AFC; and conſequently, the angle at the circumference C, which is meaſured [42] by half the arc ADB, is greater than the angle at the circumference B, which is meaſured by the half of the arc AFC.

Again, if the angle C is greater than the angle B, the ſide AB, oppoſite to the angle C, will be greater than the ſide AC, oppoſite to the angle B.

The angle C is meaſured [42] by half the arc ADB, and the angle B by half the arc AFC. But the angle C is greater than the angle B: the arc ADB is therefore greater than the arc AFC; and conſequently, the chord AB is greater than the chord AC.

PROP. XLVI.

Two parallel chords intercept equal arcs.

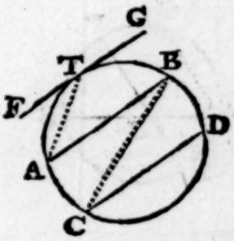


If the two chords AB, CD are parallel, the arcs AC, BD are equal. Draw the right line BC.

Becauſe the lines AB, CD are parallel, the alternate angles ABC, BCD are [17] equal. But the angle at the circumference ABC, is meaſured [42] by half the arc AC; and the angle at the circumference BCD is meaſured by half the arc BD: the arcs AC BD are therefore equal.

PROP. XLVII.

If a tangent and chord be parallel to each other, they intercept equal arcs.

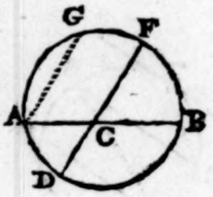


Let the tangent FG be parallel to the chord AB; the arc TA will be equal to the arc TB. Draw the right line TA.

Becauſe the lines FG, AB are parallel, the alternate angles FTA, TAB are [17] equal. But the angle FTA, formed by a tangent and a chord, is meaſured [41] by half the arc TA; and the angle at the circumference TAB is meaſured [42] by half the arc TB. The halves of the arcs TA, TB, and conſequently the arcs themſelves, are therefore equal.

PROP. XLVIII.

The angle formed by the interſection of two chords, is meaſured by half the two arcs intercepted by the two chords.



Let the two chords AB, DF cut each other at the point C; the angle FCB, or ACD, is meaſured by half the two arcs FB, AD. Draw AG parallel to DF.

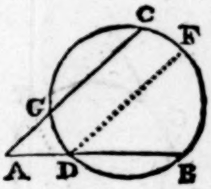
Becauſe the lines AG, DF, are parallel, the interior and exterior angles GAB, FCB, are [19] equal. But the angle at the circumference GAB, is meaſured by [42] half the arc GFB. The angle FCB is therefore alſo meaſured by half the arc GFB.

Becauſe the chords AG, DF are parallel, the arcs, GF, AD are [46] equal: AD may therefore be ſubſtituted in the room of GF; whence the angle FCB is meaſured by half the arcs AD, FB.

PROP.

PROP. XLIX.

The angle formed by two ſecants is meaſured by half the difference of the two intercepted arcs.



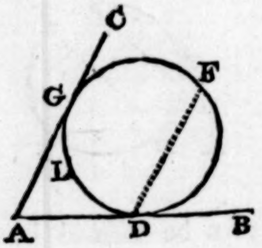
Let the angle CAB be formed by the two ſecants AC, AB, this angle is meaſured by half the difference of the two arcs GD, CB, intercepted by the two ſecants. Draw DF parallel to AC.

Becauſe the lines AC, DF are parallel, the interior and exterior angles CAB, FDB are [19] equal. But the angle FDB is meaſured [42] by half the arc FB; the angle GAB is therefore alſo meaſured by half the arc FB.

Becauſe the chords GC, DF are parallel, the arcs GD CF are [46] equal. The arc FB is therefore the difference of the arc GD and the arc CFB. Wherefore, the angle A has for its meaſure half the difference of the arcs GD, CFB.

PROP. L.

The angle formed by two tangents is meaſured by half the difference of the two intercepted arcs.

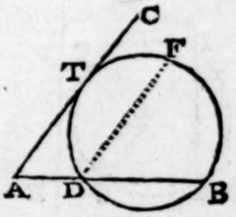


Let the angle CAB be formed by the two tangents AC, AB; this angle is meaſured by half the difference of the two arcs GLD, GFD. Draw DF parallel to AC.

Becauſe the lines AC, DF are parallel, the interior and exterior angles CAB, FDB are [19] equal. But the angle FDB, formed by the tangent DB and the chord DF, is meaſured [41] by half the arc FD. Therefore the angle CAB is alſo meaſured by half the arc FD.

Becauſe the tangent AC and the chord DF are parallel, the intercepted arcs GF, GD are [47] equal.

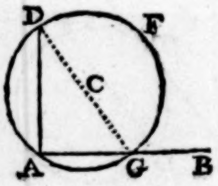
The arc FD is therefore the difference between the arc GLD and the arc GFD. Therefore the angle CAB, which is meaſured by half the arc FD, is alſo meaſured by half the difference of the arcs GLD, GFD.



COR. In the ſame manner it may be demonſtrated, that the angle formed by a tangent ATC, and a ſecant ADB, is meaſured by half the difference of the two intercepted arcs.

PROP. LI.

To raiſe a perpendicular at the extremity of a given line.



At the extremity A of the given line AB, let it be required to raiſe a perpendicular.

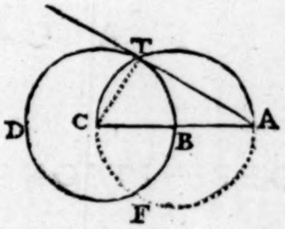
From any point C taken above the line AB, deſcribe a circumference, paſſing through the point A, and cutting the line AB in any other point, as G. Draw the diameter GD, and the right line AD; this line AD will be perpendicular to the line AB.

The angle DAG at the circumference is meaſured by [42] half the arc DFG, which is half the circumference, becauſe DCG is a diameter. The angle DAG is therefore meaſured by one fourth part of the circumference, and conſequently (def. 10.) is a right angle, whence the line AD is [11] perpendicular to the line AB.

COR. Hence it follows that the angle at the circumference which is ſubtended by a diameter, muſt be a right angle.

PROP. LII.

From any point without a circle to draw a tangent to that circle.



From the point A let it be required to draw a tangent to the circle DTB.

Draw, from the center C, any right line CA; biſect this right line; and from the point of diviſion B as a center deſcribe the arc CTA. Laſtly, from the point A, and through the point T, in which the two arcs cut each other, draw the right line AT; this right line AT will be a tangent to the circle DTB. Draw the radius CT.

The angle CTA at the circumference, being ſubtended by the diameter CA, is (51 cor.) a right angle; therefore the line TA is perpendicular to the extremity of the radius CT; and conſequently 4° is a tangent to the circle DTB.

OF THE

MENSURATION

OF

SURFACES.

DEFINITIONS.

1. A *Mathematical Point* has neither length, breadth, nor thickneſs. The *Phyſical Point*, of which we are now to ſpeak, has a ſuppoſed length and breadth exceedingly ſmall.

2. A *Phyſical Line* is a ſeries of phyſical points; and conſequently, its breadth is equal to that of the phyſical points of which it is compoſed.

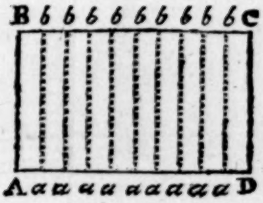
3. Since phyſical lines are compoſed of points, as numbers are compoſed of units, points may be called the *Units* of lines.

4. As, to multiply one number by another, is to take or repeat the firſt number as many times as there are units in the ſecond; ſo, to *multiply one line by another*, is to take or repeat the firſt line as many times as there are units, that is phyſical points, in the ſecond.

PROP.

PROP. LIII.

The ſurface of a rectangle is equal to the product of its two ſides.



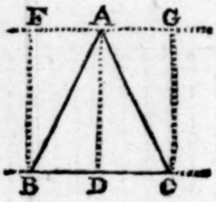
Let the rectangle be ABCD. If the phyſical line AB be multiplied by the phyſical line AD, the product will be the ſurface ABCD.

If as many phyſical lines, equal to AB, as there are phyſical points in the line AD, be raiſed perpendicularly upon AD, theſe lines, AB, *ab*, &c. will fill up the whole ſurface of the rectangle ABCD. Therefore the ſurface ABCD is equal to the line AB taken as many times as there are phyſical points in the line AD, that is, (def. 4.) equal to the line AB multiplied by the line AD.

PROP.

PROP. LIV.

The ſurface of a triangle is equal to half the product of its altitude and baſe.



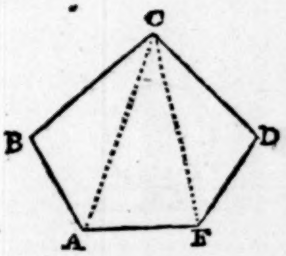
If from the vertex of any angle, A, of the triangle BAC, be drawn AD perpendicular to the oppoſite ſide BC, this perpendicular is called the *height*, and the ſide BC the *baſe*, of the triangle. Now, the ſurface of the triangle is equal to half the product of the height AD and the baſe BC.

Produce BC both ways; through the point A, draw FG parallel to BC; and raiſe the two perpendiculars BF, CG.

Becauſe the rectangle BFGC and the triangle BAC are between the ſame parallels and have the ſame baſe, the triangle is 29 half the rectangle. But the ſurface of the rectangle is equal 53 to the product of BF and BC. Therefore the ſurface of the triangle is equal to half the product of BF and BC, that is, of DA and BC.

PROP. LV.

To meaſure the ſurface of any rectilineal figure.

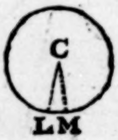


Let ABCDFA be the rectilineal figure, of which it is required to find the ſurface.

Divide the whole figure into triangles by drawing the lines CA, CF. Then drawing a perpendicular from the point B to the ſide CA, multiply theſe two lines : the half of their product will 54 give the ſurface of the triangle ABC. In the ſame manner let the ſurfaces of the remaining triangles ACF, FCD be found. Theſe three ſurfaces added together will give the whole ſurface of the figure ABCDFA.

PROP. LVI.

The area of a circle is equal to half the product of its radius and circumference.



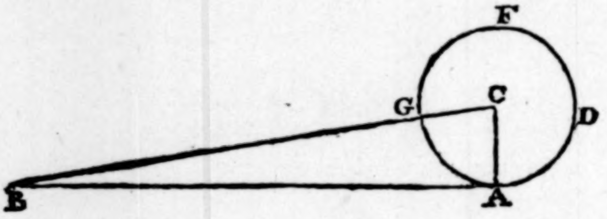
If the radius of the circle C be multiplied by its circumference, the half of the product will give the ſurface of the circle.

Two phyſical points being manifeſtly not ſufficient to make a curve line, this muſt require at leaſt three. If therefore all the phyſical points of a circumference be taken two by two, theſe will compoſe a great number of ſmall right lines. From the extremities L, M, of one of theſe ſmall right lines, if two radii LC, MC be drawn, a ſmall triangle LCM will be formed, the ſurface of which will be equal to half the product of its height, that is the radius, and its baſe.

To find the ſurface of all the ſmall triangles of which the circle is compoſed, multiply the height, that is the radius, by all the baſes, that is by the circumference, and take the half of the product: whence the area or ſurface of the circle will be equal to half the product of the radius and circumference.

PROP. LVII.

To draw a triangle equal to a given circle.



Let it be required to form a triangle, the ſurface of which ſhall be equal to that of the circle AGFDA.

At the extremity of any radius of the circle, CA, raiſe a perpendicular AB equal to the circumference AGFD, and draw the right line CB; the ſurface of the triangle BCA will be equal to that of the circle AGFDA.

The ſurface of the circle is equal 56 to half the product of the radius CA and the circumference or the line AB. The ſurface of the triangle is alſo equal 54 to half the product of its height CA, or radius, and its baſe BA, or the circumference. Therefore the ſurface of the triangle is equal to that of the circle.

OF

OF

PROPORTION.

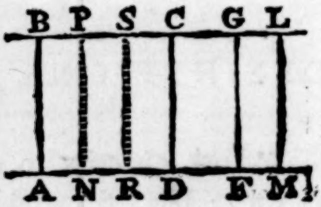
DEFINITIONS.

1. THE *Ratio* of one quantity to another is the number of times which the firſt contains the ſecond: thus the ratio of 12 to 3 is four, becauſe 12 contains 3 four times; "or more univerſally, *ratio* is the comparative magnitude of one quantity with reſpect to another."

2. Four quantities are *proportionals*, or *in geometrical proportion*, or "two quantities are ſaid to have the ſame ratio with two others," when the firſt contains, or is contained in, the ſecond, exactly the ſame number of times which the third contains, or is contained in, the fourth. Thus the four numbers, 6, 3, 8, 4 are proportionals, becauſe 6 contains 3 as many times as 8 contains 4, and 3 is contained in ſix as many times as 4 is contained in 8, that is twice; which is thus expreſſed, 6 is to 3 as 8 to 4, or 3 is to 6 as 4 to 8.

PROP. LVIII.

Parallelograms which are between the ſame parallels, are to one another as their baſes.



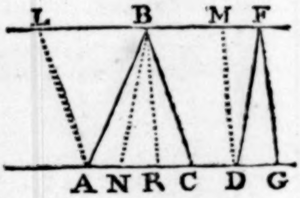
Let the two parallelograms ABCD, FGLM, be between the ſame parallels BL, AM; the ſurface of the parallelogram ABCD, contains the ſurface of the parallelogram FGLM, as many times, exactly, as the baſe AD contains the baſe FM. Suppoſe, for example, that the baſe AD is triple of the baſe FM; in this caſe the ſurface ABCD will alſo be triple of the ſurface FGLM.

Divide the baſe AD into three parts, each of which is equal to the baſe FM; and draw from the points of diviſion the lines NP, RS parallel to the ſide AB.

The parallelograms ABPN, FGLM, being between the ſame parallels and having equal baſes, the parallelogram ABPN is 3° equal to the parallelogram FGLM. For the ſame reaſon, the parallelograms NPSR, RSCD are alſo equal to the parallelogram FGLM. The parallelogram ABCD is therefore compoſed of three parallelograms, each of which is equal to the parallelogram FGLM. Conſequently the parallelogram ABCD is triple of the parallelogram FGLM.

PROP. LIX.

Triangles which are between the ſame parallels are to one another as their baſes.



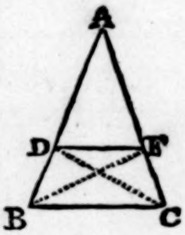
Let the two triangles ABC, DFG be between the ſame parallels LF, AG; the ſurface of the triangle ABC contains the ſurface of the triangle DFG as many times as the baſe AC contains the baſe DG. Suppoſe, for example, that the baſe AC is triple of the baſe DG; in this caſe the ſurface ABC will be triple of the ſurface DFG.

Divide the baſe AC into three equal parts AN, NR, RC, each of which is equal to the baſe DG; and draw the right lines BN, BR.

The triangles ABN, DFG, being between the ſame parallels, and having equal baſes, the triangle ABN is [31] equal to the triangle DFG. For the ſame reaſon the triangles NBR, RBC, are each equal to the triangle DFG. The triangle ABC is therefore compoſed of three triangles, each of which is equal to the triangle DFG. Therefore the triangle ABC is triple of the triangle DFG.

PROP. LX.

If a line be drawn in a triangle parallel to one of its ſides, it will cut the other two ſides proportionally.



In the triangle BAC, if the line DF is parallel to the ſide BC, it will cut the other two ſides in ſuch manner, that the ſegment AD will be to the ſegment DB, as the ſegment AF is to the ſegment FC. Suppoſe, for example, the ſegment AD to be triple of the ſegment DB; the ſegment AF will be triple of the ſegment FC. Draw the two diagonals DC, FB.

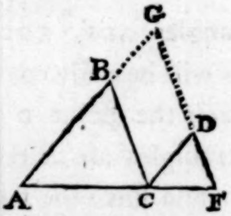
The triangles AFD, DFB are between the ſame parallels; as will be eaſily conceived by ſuppoſing a line drawn through the point F parallel to the ſide AB. Theſe two triangles are therefore to one another 59 as their baſes; and ſince the baſe AD is triple of the baſe DB, the triangle AFD will be triple of the triangle DFB.

Again, the triangles BFD, FDC are between the ſame parallels DF, BC, and upon the ſame baſe DF. Theſe two triangles are therefore 31 equal, and ſince the triangle AFD is triple of the triangle DFB, it will alſo be triple of the triangle FDC.

Laſtly, the triangles ADF, FDC are between the ſame parallels; as will be eaſily conceived by ſuppoſing a line drawn through the point D parallel to the ſide AC. Theſe two triangles are therefore to one another 59 as their baſes: and ſince the triangle ADF is triple of the triangle FDC, the baſe AF will be triple of the baſe FC.

PROP. LXI.

Equiangular triangles have their homologous ſides proportional.

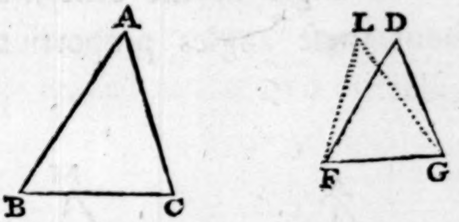


In the two triangles ABC, CDF, if the angle A is equal to the angle C, the angle B equal to the angle D, and the angle C equal to the angle F; the ſide AC, for example, oppoſite to the angle B, is to the ſide CF oppoſite to the angle D, as the ſide AB oppoſite to the angle C, is to the ſide CD oppoſite to the angle F. Place the two triangles ſo that the ſides AC, CF ſhall form one right line; and produce the ſides AB, FD, till they meet in G.

The interior and exterior angles GAF, DCF, being equal, the lines GA, DC are [20] parallel. In like manner the alternate angles on the ſame ſide GFA, BCA, being equal, the lines GF, BC are [20] parallel. Whence the quadrilateral figure BGDC is a parallelogram, and conſequently, its oppoſite ſides are equal. In the triangle GAF, the line BC being parallel to the ſide GF cuts [60] the other two ſides proportionally; that is, AC is to CF, as AB is to BG, or its equal CD.

PROP. LXII.

Triangles, the ſides of which are proportional, are equiangular.



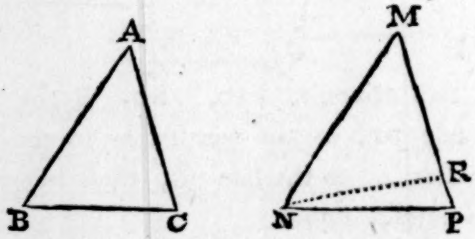
In the two triangles BAC, FDG, if the ſide AB is to the ſide DF, as the ſide BC is to the ſide FG, and as the ſide AC to the ſide DG, theſe two triangles have their angles equal.

Let the ſide AB be ſuppoſed triple of the ſide DF; the ſide AC muſt be triple of the ſide DG, and the ſide BC triple of the ſide FG.

If the triangle FDG be not equiangular with the triangle BAC, another triangle may be formed equiangular with it, for example FLG. But this is impoſſible: for if the two triangles BAC, FLG were equiangular, their ſides would be [61] proportional, and BC being triple of FG, AB would be triple of LF: but AB is triple of DF; whence LF would be equal to DF. For the ſame reaſon LG would be equal to DG. Thus the two triangles FLG, FDG, having their three ſides equal would be [5] identical; which is abſurd, ſince their angles are unequal.

PROP. LXIII.

Triangles which have an angle in one equal to an angle in the other, and the ſides about theſe angles proportional, are equiangular.



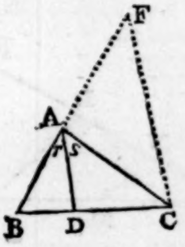
If, in the two triangles BAC, NMP, the angle A be equal to the angle M, and the ſide AB be to the ſide MN as the ſide AC is to the ſide MP, the two triangles are equiangular.

If AB be triple of MN, AC muſt be triple of MP. Now, if the angle MNP, for example, is not equal to the angle ABC, another angle may be made, as MNR, which ſhall be equal to it. But this is impoſſible: for, the two triangles BAC, NMR having two angles equal, would be equiangular, and conſequently 61 would have their ſides proportional: whence AB being triple of MN, AC would be triple of MR; which cannot be, ſince AC is triple of MP.

PROP.

PROP. LXIV.

A right line which biſects any angle of a triangle, divides the ſide oppoſite to the biſected angle into two ſegments, which are proportional to the two other ſides.



In the triangle BAC, let the angle BAC be biſected by the right line AD, making the angle r equal to the angle s. The ſegment BD is to the ſegment DC, as the ſide BA to the ſide AC.

Produce the ſide BA, and draw CF parallel to DA.

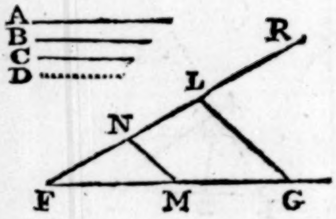
The lines DA, CF being parallel, the interior and exterior angles, r, F, are [19] equal, and the alternate angles, s, C, are [17] alſo equal. And ſince the angle r is equal to the angle s, the angle F will alſo be equal to the angle C; and conſequently the ſide AF is equal to the ſide AC.

In the triangle BFC, the line AD being parallel to the ſide FC; BD [60] will be to DC, as BA is to AF or its equal AC.

PROP.

PROP. LXV.

To find a fourth proportional to three given lines.



Let the three lines be A, B, C; it is required to find a fourth line D, ſuch that the line A ſhall be to the line B, as the line C is to the line D.

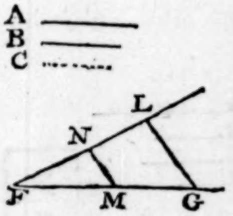
Form any angle RFG; make FM equal to the line A, MG equal to the line B, and FN equal to the line C; draw the right line MN, and through the point G, draw GL parallel to MN; NL will be the fourth proportional required.

In the triangle FLG, the line NM, being parallel to the ſide LG, cuts the other two ſides 6° proportionally. Whence FM is to MG, as FN is to NL; that is A is to B, as C is to D.

PROP.

PROP. LXVI.

To find a third proportional to two given lines.



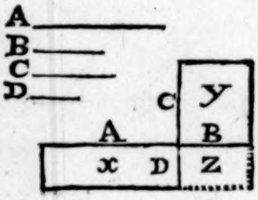
Let the two lines be A, B; it is required to find a third line C, ſuch that the line A ſhall be to the line B, as the line B is to the line C.

Form any angle LFG; make FM equal to the line A, MG equal to the line B, and FN equal to the line B: draw the right line MN, and through the point G draw GL parallel to MN; NL will be the third proportional required.

In the triangle FLG, the line NM being parallel to the ſide LG, cuts the other two ſides [60] proportionally. Whence FM is to MG as FN is to NL; that is, A is to B, as B is to C.

PROP. LXVII.

If four lines be proportional, the rectangle or product of the extremes, is equal to the rectangle or product of the means.



Let the line A be to the line B, as the line C is to the line D; the rectangle formed by the lines A and D, is equal to the rectangle formed by the lines B and C.

Let the four lines meet in a common point, forming at that point four right angles; and draw lines parallel to them to complete the rectangles *x*, *y*, *z*.

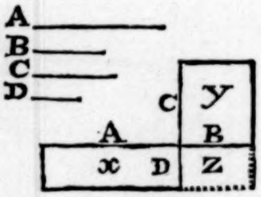
If the line A be triple of the line B, the line C will be triple of the line D.

The rectangles or parallelograms *x*, *z*, being between the ſame parallels, are to one another as their baſes. Since the baſe A is triple of the baſe B, the rectangle *x* is therefore triple of the rectangle *z*. In like manner, the rectangles or parallelograms *y*, *z*, being between the ſame parallels, are to one another as their baſes: ſince the baſe C is triple of the baſe D, the rectangle *y* is therefore triple of the rectangle *z*. Wherefore, the rectangle *x* being triple of the rectangle *z*, and the rectangle *y* being triple of the ſame rectangle *z*, theſe two rectangles *x*, and *y*, are equal to one another.

PROP.

PROP. LXVIII.

Four lines, which have the rectangle or product of the extremes equal to the rectangle or product of the means, are proportional.



Let the four lines A, B, C, D, be ſuch that the rectangle of A and D is equal to the rectangle of B and C; the line A will be to the line B as the line C to the line D.

Let the four lines meet in a common point, forming at that point four right angles; and complete the rectangles *x*, *y*, *z*.

If the line A be triple of the line B, the line C will be triple of the line D.

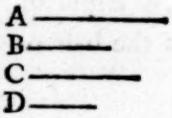
The rectangles *x* and *z*, being between the ſame parallels, are to one another as their baſes. Since the baſe A is triple of the baſe B, the rectangle *x* will therefore be triple of the rectangle *z*. And the rectangle *y* is by ſuppoſition equal to the rectangle *x*: the rectangle *y* is therefore alſo triple of the rectangle *z*.

But the rectangles *y*, *z*, being between the ſame parallels, are to one another as their baſes: therefore, ſince the rectangle *y* is triple of the rectangle *z*, the baſe C is alſo triple of the baſe D.

PROP.

PROP. LXIX.

If four lines are proportional, they are alſo proportional alternately.



If the line A is to the line B as the line C to the line D; they will be in proportion *alternately*, that is, the line A will be to the line C as the line B to the line D.

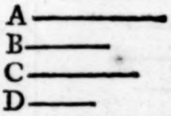
Becauſe the line A is to the line B as the line C is to the line D, the rectangle of the extremes, A and D is equal to the rectangle of the means B and C: whence it follows [68] that the line A is to the line C, as the line B to the line D.

Otherwiſe:

Suppoſing the line A to be triple of the line B, the line C will be triple of the line D. Hence inſtead of ſaying A is to B as C to D, we may ſay, three times B is to B as three times D is to D. Now it is manifeſt that three times B is to three times D, as B is to D. Therefore the line A (which is equal to three times B) is to the line C (which is equal to three times D) as the line B is to the line D.

PROP. LXX.

If four lines are proportional, they will be proportional by compoſition.

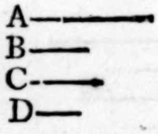


Let the line A be to the line B as the line C is to the line D, they will be proportional by *compoſition*; that is, the line A joined to the line B, will be to the line B, as the line C joined to the line D, is to the line D.

If the line A contains the line B, for inſtance, three times, and the line C contains the line D three times; the line A joined to the line B, will contain the line B four times; and the line C joined to the line D, will contain the line D four times. Therefore the line A joined to the line B, is to the line B as the line C joined to the line D, is to the line D.

PROP. LXXI.

If four lines are proportional, they will alſo be proportional by diviſion.



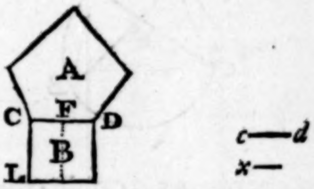
If the line A is to the line B as the line C is to the line D, they will be proportional by *diviſion*, that is, the line A wanting the line B, is to the line B, as the line C wanting the line D, is to the line D.

If the line A contains the line B, for example, three times, and the line C contains the line D three times; the line A wanting the line B, will contain the line B only twice: and the line C wanting the line D, will alſo contain the line D twice. Therefore the line A wanting the line B, is to the line B, as the line C wanting the line D, is to the line D.

PROP.

PROP. LXXII.

If three lines are proportional, the firſt is to the third, as the ſquare of the firſt is to the ſquare of the ſecond.



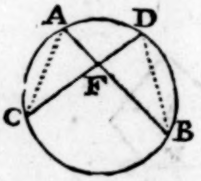
If the line CD is to the line *cd* as the line *cd* is to a third line *x*; the line CD is to the line *x*, as the ſquare of the line CD is to the ſquare of the line *cd*. Take CF equal to the line *x*, and draw the perpendicular FB.

Since the line CD is to the line *cd* as the line *cd* is to the line CF, the rectangle of the extremes CF, CD or CL, is equal [67] to the rectangle of the means, that is to the ſquare of *cd*.

Again, the ſquare of CD, and the rectangle of the lines CF, CL, being between the ſame parallels, are to one another [58] as their baſes. Therefore, CD is to CF, or *x*, as the ſquare of CD is to the rectangle of CF and CL, or to its equal the ſquare of *cd*.

PROP. LXXIII.

If two chords in a circle cut each other, the rectangle of the ſegments of one is equal to the rectangle of the ſegments of the other.

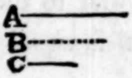


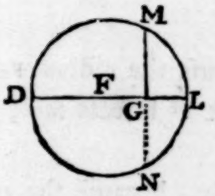
Let the two chords AB, CD in the circle cut each other in the point F, the rectangle of AF, FB is equal to the rectangle of CF, FD. Draw the two right lines AC, DB.

Becauſe in the triangles CAF, BDF the angles at the circumference A and D are both meaſured [42] by half the arc CB, they are equal. Becauſe the angles C and B are both meaſured [42] by half the arc AD, theſe angles are alſo equal. And the angles at F are equal becauſe they are vertical. Theſe two triangles are therefore equiangular, and conſequently, [61] their ſides are proportional. Whence the ſide AF oppoſite to the angle C, is to the ſide FD oppoſite to the angle B, as the ſide CF oppoſite to the angle A, is to the ſide FB oppoſite to the angle D. Therefore [69] the rectangle of the extremes AF, FB is equal to the rectangle of the means CF, FD.

PROP. LXXIV.

To find a mean proportional between two given lines.

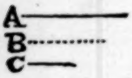


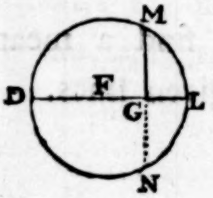


Let there be two lines A, C; it is required to find a third line B, ſuch that the line A ſhall be to the line B, as the line B is to the line C.

Place the lines A and C in ſuch manner that they ſhall form one right line DGL; and biſect this right line in the point F. From the point F as a center, deſcribe the circumference of a circle DMLN; then, at at the point G, where the two lines are joined, raiſe the perpendicular GM; GM is the mean proportional ſought between the lines A and C. Produce MG to N.

Becauſe the chords DL, MN cut each other at the point G, the rectangle of the ſegments DG, GL is 73 equal to the rectangle of the ſegments MG, GN.





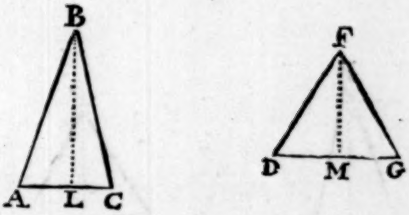
Becauſe the radius FL is perpendicular to the chord MN, FL [38] biſects MN; therefore GN is equal to GM.

Laſtly, becauſe the rectangle of the extremes DG, GL is equal to the rectangle of the means GM, GN or its equal GM, DG is to GM, as GM is to GL. Therefore GM is a mean proportional between DG and GL, that is, between the lines A and C.

PROP.

PROP. LXXV.

The baſes and altitudes of equal triangles are in reciprocal or inverſe ratio.



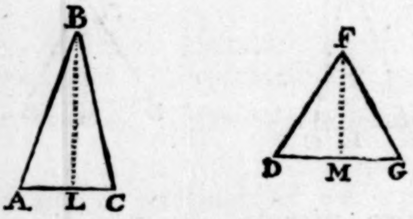
Let the two triangles ABC, DFG be equal: the baſe AC will be to the baſe DG, as the perpendicular FM to the perpendicular BL, that is, the baſes and altitudes are in *reciprocal* or *inverſe ratio.*

The triangle ABC is 54 half the product or rectangle of the baſe AC and the altitude BL. Again, the triangle DFG is 54 half the product or rectangle of the baſe DG and the altitude FM. The two triangles being equal, the two rectangles, which are double of the triangles will therefore alſo be equal.

Again, becauſe the rectangle of the extremes AC, BL, is equal to the rectangle of the means DG, FM, 68 AC is to DG as FM is to BL.

PROP. LXXVI.

Triangles, the baſes and altitudes of which are in reciprocal or inverſe ratio, are equal.



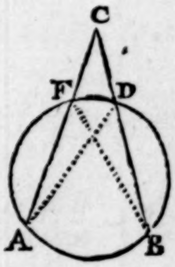
In the two triangles ABC, DFG, if the baſe AC is to the baſe DG, as the perpendicular FM to the perpendicular BL, the ſurfaces of the two triangles are equal.

Becauſe AC is to DG as FM is to BL, the product or rectangle of the extremes AC, BL is 67 equal to the product or rectangle of the means DG, FM. The halves (27. Cor.) of theſe two rectangles, namely, the triangles ABC, DFG, are therefore equal.

PROP.

PROP. LXXVII.

Two ſecants drawn from the ſame point to a circle, are in the inverſe ratio of the parts which lie out of the circle.



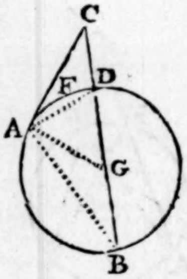
Let the two ſecants be CA, CB; CA is to CB, as CD is to CF. Draw the right lines FB, DA.

In the triangles CDA, CFB, the angles at the circumference A and B, being both meaſured [42] by half the arc FD, are equal: and the angle C is common to the two triangles. Theſe two triangles are therefore [23] equiangular, and [61] have their ſides proportional. Whence the ſide CA of the firſt triangle, is to the ſide CB of the ſecond triangle, as the ſide CD of the firſt triangle is to the ſide CF of the ſecond triangle.

PROP.

PROP. LXXVIII.

The tangent to a circle is a mean proportional between the ſecant, and the part of the ſecant which lies out of the circle.



In the circle ABD, CB being ſecant and CA tangent; CB is to CA, as CA is to CD. Draw the right lines AB, AD.

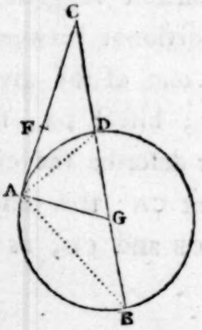
The triangles CAB, CDA have the angle C common to both. Alſo, the angle B is meaſured [42] by half the arc AFD; and the angle CAD, formed by the tangent AC and the chord AD, is meaſured [41] by half the ſame arc AFD. The two triangles CAB, CDA, having then two angles equal, are [23] equiangular, and conſequently [61] have their ſides proportional. Hence the ſide CB of the greater triangle, oppoſite to the angle CAB, is to the ſide CA of the ſmaller triangle oppoſite to the angle D, as the ſide CA of the greater triangle, oppoſite to the angle B, is to the ſide CD of the ſmaller triangle, oppoſite to the angle A.

COR. This propoſition ſuggeſts a new method of finding a mean proportional between two given lines. Take CB equal to one of the given lines, and CD equal to the other; biſect DB; from the point of diviſion as a center deſcribe the circumference DAB; and draw the tangent CA: this tangent is a mean proportional between CB and CD, as appears from the propoſition.

PROP.

PROP. LXXIX.

To cut a given line in extreme and mean ratio.



Let it be required to divide the line CA in extreme and mean ratio: that is, to divide it in ſuch manner, that the whole line ſhall be to the greater part, as the greater part is to the leſs.

At the extremity A of the line CA, raiſe a perpendicular AG, equal to half the line CA: from the point G as a center, with the radius GA, deſcribe the circumference ADB; draw the line CB through the center; and take CF equal to CD: the line CA will be divided, at the point F in extreme and mean ratio.

Becauſe 78 CB is to CA, as CA is to CD, by diviſion 71 CB wanting CA, or its equal DB, is to CA, as CA wanting CD, or its equal CF, is to CD; that is, CD or CF is to CA as FA is to CD or CF: that is inverſely, CA is to CF, as CF is to FA, or the line AC is cut in extreme and mean ratio.

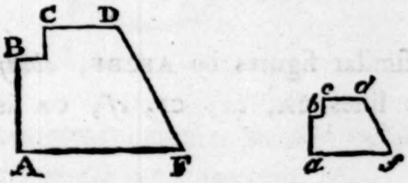
OF

OF

SIMILAR FIGURES.

DEFINITIONS.

1. FIGURES are *Similar*, which are compoſed of an equal number of phyſical points diſpoſed in the ſame manner. Thus the figures ABCDF, *abcdf* are ſimilar, if every point in the firſt figure has its correſponding point, placed in the ſame manner, in the ſecond.



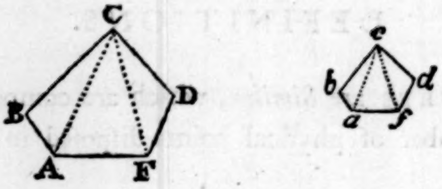
Hence it follows, that if the firſt figure is, for example, three times greater than the ſecond, the points of which it is compoſed are three times greater than thoſe of the ſecond figure.

2. In ſimilar figures, thoſe lines are ſaid to be *homologous* which are compoſed of an equal number of correſponding points.

PROP.

PROP. LXXX.

In ſimilar figures, the homologous lines are proportional.

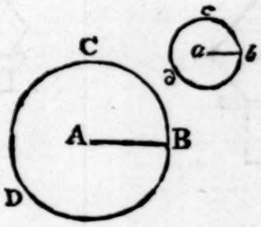


Let the ſimilar figures be ABCDF, *abcdf*, and the homologous lines CA, *ca*; CF, *cf*; CA is to CF, as *ca* is to *cf*.

Since the lines CA, *ca* are homologous, they are compoſed of an equal number of correſponding points; as are alſo the homologous lines CF, *cf*. If, for example, the line CA is compoſed of 40 equal points, and the line CF of 30; the line *ca* will neceſſarily be compoſed of 40 points, and the line *cf* of 30; and it is manifeſt that 40 is to 30, as 40 to 30. Therefore CA is to CF as *ca* to *cf*.

PROP. LXXXI.

The circumferences of circles are as their radii.



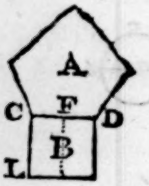
The circumference DCB is to the radius AB, as the circumference *dcb* is to the radius *ab*.

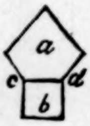
All circles are ſimilar figures, that is, are compoſed of an equal number of points, diſpoſed in the ſame manner: they have therefore 80 their homologous lines proportional. Therefore the circumference DCB is to the radius AB, as the circumference *dcb* is to the radius *ab*.

PROP.

PROP. LXXXII.

Similar figures are to each other as the ſquares of their homologous ſides.





Let the two ſimilar figures be A, *a*. Upon the homologous ſides CD, *cd* form the ſquares B, *b*. The ſurface A is to the ſurface *a*, as the ſquare B is to the ſquare *b*.

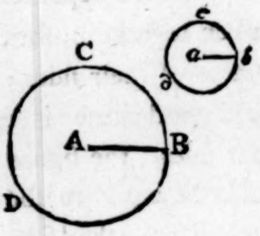
Since the figures A, *a* are ſimilar, they are compoſed of an equal number of correſponding points; and ſince the homologous ſides CD, *cd*, are compoſed of an equal number of points, the ſquares drawn upon theſe lines, B, *b*, are alſo compoſed of an equal number of points.

If it be ſuppoſed that the ſurface A is compoſed of 1000 points, and the ſquare B of 400 points; the ſurface *a* will alſo be compoſed of 1000 points and the ſquare *b* of 400. Now it is manifeſt that 1000 is to 400, as 1000 to 400. Therefore the ſurface A is to the ſquare B as the ſurface *a* is to the ſquare *b*; and alternately 69 the ſurface A is to the ſurface *a*, as the ſquare B to the ſquare *b*.

COR. Hence it follows, that if any three ſimilar figures be formed upon the three ſides of a right-angled triangle, the figure upon the hypothenuſe will be equal to the other two taken together: for theſe three figures will be as the ſquares of their ſides; therefore, ſince the ſquare of the hypothenuſe is equal to the two ſquares of the other ſides, the figure formed upon the hypothenuſe will alſo be equal to the two other ſimilar figures formed upon the other ſides.

PROP. LXXXIII.

Circles are to each other as the ſquares of their radii.



Let two circles, DCB, *dcb*, be drawn.

The ſurface contained within the circumference DCB, is to the ſurface contained within the circumference *dcb*, as the ſquare formed upon the radius AB to the ſquare formed upon the radius *ab*.

The two circles, being ſimilar figures, are compoſed of an equal number of correſponding points. And the radii AB, *ab*, being compoſed of an equal number of points, the ſquares of theſe radii will alſo be compoſed of an equal number of points.

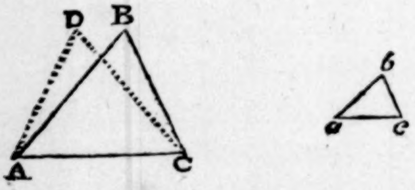
Suppoſe, for example, that the greater circle DCB is compoſed of 800 points, and the ſquare of the greater radius AB of 300 points, the ſmaller circle *dcb* will alſo be compoſed of 800 points, and the ſquare of the ſmaller radius of 300. Now it is manifeſt that 800 is to 300 as 800 to 300. Therefore the greater circle DCB is to the ſquare of its radius AB, as the ſmaller circle *dcb* is to the ſquare of its radius *ab*; and alternately, the greater circle is to the leſſer circle, as the greater ſquare is to the leſſer ſquare.

PROP. LXXXIV.

Similar triangles are equiangular.

If the two triangles ABC, *abc* be compoſed of an equal number of points diſpoſed in the ſame manner, they are equiangular.

For, ſince the triangles ABC, *abc* are ſimilar figures, they have their ſides [8•] proportional: they are therefore [62] equiangular.



PROP. LXXXV.

Equiangular triangles are ſimilar.

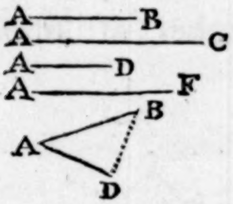
If the triangles ABC, *abc*, are equiangular, they are alſo ſimilar.

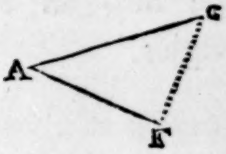
If the triangle ABC were not ſimilar to the triangle *abc*, another triangle might be formed upon the line AC, for example ADC, which ſhould be ſimilar to the triangle *abc*. Now the triangle ADC being ſimilar to the triangle *abc*, will alſo [84] be equiangular to *abc*, which is impoſſible, ſince the triangle ABC is ſuppoſed equiangular to *abc*.

PROP.

PROP. LXXXVI.

If four lines are proportional, their ſquares are alſo proportional.





If the line AB is to the line AC, as the line AD is to the line AF; the ſquare of the line AB will be to the ſquare of the line AC, as the ſquare of the line AD is to the ſquare of the line AF.

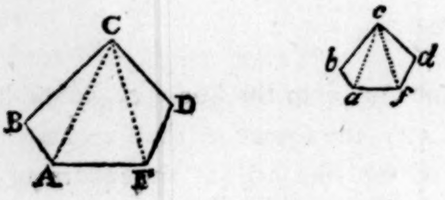
With the lines AB and AD form an angle BAD; with the lines AC and AF form another angle CAF equal to the angle BAD; and draw the right lines BD, CF.

Becauſe AB is to AC as AD to AF, and the contained angles are equal, the two triangles BAD, CAF have their ſides about equal angles proportional: they are therefore [63] equiangular, and conſequently [85] ſimilar. Whence they are to one another [82] as the ſquares of their homologous ſides. If then the triangle BAD be a third part of the triangle CAF, the ſquare of the ſide AB will be a third part of the ſquare of the ſide AC, and the ſquare of the ſide AD will be a third part of the ſquare of the ſide AF. Therefore theſe four ſquares will be proportional.

PROP.

PROP. LXXXVII.

Similar rectilineal figures may be divided into an equal number of ſimilar triangles.



Let the ſimilar figures be ABCDF, *abcdf*; and draw the homologous lines CA, *ca*; CF, *cf*; theſe two figures will be divided into an equal number of ſimilar triangles.

The triangles BCA, *bca*, being compoſed of an equal number of correſponding points, are ſimilar. The triangles ACF, *acf*; and the triangles FCD, *fcd* are alſo ſimilar for the ſame reaſon. Therefore the ſimilar figures ABCDF, *abcdf*, are divided into an equal number of ſimilar triangles.

PROP.

PROP. LXXXVIII.

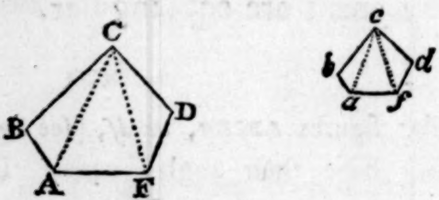
Similar figures are equiangular.

The ſimilar figures ABCDF, *abcdf*, [ſee fig. on the oppoſite ſide] have their angles equal. Draw the homologous lines CA, *ca*; CF, *cf*.

The triangles BCA, *bca* are ſimilar, and conſequently equiangular. Therefore the angle B is equal to the angle *b*, the angle BAC to the angle *bac*, and the angle BCA to the angle *bca*. The triangles ACF, *acf*; FCD, *fcd*, are alſo equiangular becauſe they are ſimilar. Therefore all the angles of the ſimilar figures ABCDF, *abcdf* are equal.

PROP. LXXXIX.

Equiangular figures, the ſides of which are proportional, are ſimilar.



If the figures ABCDF, *abcdf* have their angles equal, and their ſides proportional, they are ſimilar. Draw the right lines CA, *ca*; CF, *cf*.

The triangles CBA, *cba* have two ſides proportional and the contained angle equal; they are therefore [63] equiangular, and conſequently [85] ſimilar. The lines CA, *ca* are therefore [80] proportional.

The triangles CAF, *caf* have two ſides proportional, and the contained angle equal; for, if from the equal angles BAF, *baf* be taken the equal angles BAC, *bac*, there will remain the equal angles CAF, *caf*. Theſe two triangles are therefore equiangular, and conſequently ſimilar. In the ſame manner it may be proved, that the triangles CFD, *cfd* are ſimilar.

The two figures ABCDF, *abcdf* are then compoſed of an equal number of ſimilar triangles; that is, they are compoſed of an equal number of points diſpoſed in the ſame manner, or are ſimilar.

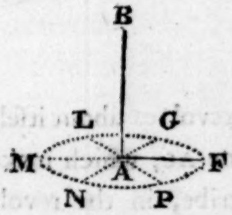
PROP.

OF PLANES.

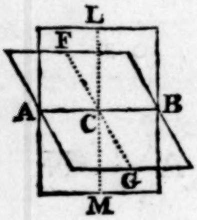
DEFINITIONS.

1. A PLANE is a ſurface ſuch that if a right line, applied to it, touches it in two points, it will touch it in every other point. The ſurface of a fluid at reſt, or of a well-poliſhed table, may be conſidered as a Plane.

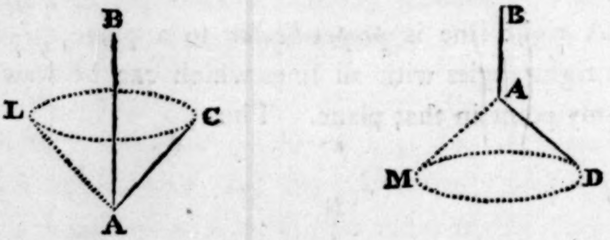
2. A right line is *perpendicular* to a plane, if it makes right angles with all lines which can be drawn from any point in that plane. Thus



BA is perpendicular to the plane MLGFPN, becauſe it makes right angles with the lines AM, AL, AG, &c. drawn from the point A.

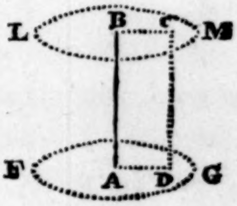


3. Let AB be the common interſection of two planes. If two right lines LM, FG be drawn in theſe two planes perpendicular to the line AB, theſe will form four angles at the point C, which are called the *Inclinations* of the two planes, or the angles formed by the two planes.



4. If the line AB revolves about itſelf without changing its place, the line AC, which makes an acute angle with AB, will deſcribe, in the revolution, a concave ſurface LAC; and the line AD, which makes an obtuſe angle with AB, will deſcribe in the revolution a convex ſurface, MAD.

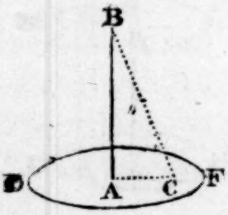
5. But the line AF, [ſee fig. def. 2.] which makes a right angle with AB, will deſcribe in the revolution a ſurface which will be neither concave nor convex, but plane: and the line AB will be perpendicular to the plane MLGFPN, becauſe it will make right angles with the lines AM, AL, AG, &c. drawn from the point A in that plane.



6. Two planes are *Parallel*, when all perpendiculars drawn from one to the other are equal.

PROP. XC.

A perpendicular is the ſhorteſt line which can be drawn from any point to a plane.



From the point B, let the right line BA be drawn perpendicular to the plane DF; any other line, as BC, will be longer than the line BA. Upon the plane draw the right line AC.

Becauſe the line BA is perpendicular to the plane DF, the angle BAC is a right angle. The ſquare of BC is therefore [32] equal to the ſquares of BA and AC taken together. Conſequently, the ſquare of BC is greater than the ſquare of BA, and the line BC longer than the line BA.

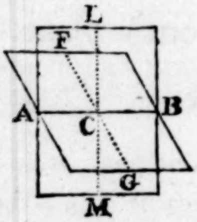
PROP. XCI.

A perpendicular meaſures the diſtance of any point from a plane.

The diſtance of one point from another is meaſured by a right line, becauſe this is the ſhorteſt line which can be drawn from one point to another. So, the diſtance from a point to a line, is meaſured by a perpendicular, becauſe this line is the ſhorteſt which can be drawn from the point to the line. In like manner, the diſtance from a point to a plane, muſt be meaſured by a perpendicular, drawn from that point to the plane, becauſe this is the ſhorteſt line which can be drawn from the point to the plane.

PROP. XCII.

The common interſection of two planes is a right line.

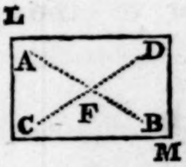


Let the two planes ALBMA, AFBGA interſect each other; the line which is common to both is a right line. Draw a right line from the point A to the point B.

Becauſe the right line AB touches the two planes in the points A and B, it will touch them [def. 1.] in all other points: this line is therefore common to the two planes. Wherefore the common interſection of the two planes is a right line.

PROP. XCIII.

If three points, not in a right line, are common to two planes, theſe two planes are one and the ſame plane.



Let two planes be ſuppoſed to be placed upon one another, in ſuch manner that the three points, A, B, C ſhall be common to the two planes; all their other points will alſo be common, and the two planes will be one and the ſame plane. The point D, for example, is common to both planes. Draw the right lines AB, CD.

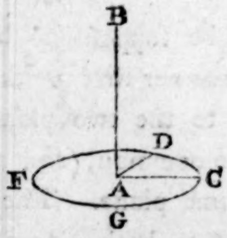
Becauſe the right line AB touches the two planes in the points A and B, it will touch them [def. I.] in every other point; it will therefore touch them in the point F. The point F is therefore common to the two planes.

Again, becauſe the right line CD touches the two planes in the points C and F, it will touch it in the point D: therefore the point D is common to the two planes. The ſame may be ſhewn concerning every other point. Wherefore the two planes coincide in all points, or are one and the ſame plane.

PROP.

PROP. XCIV.

If a right line be perpendicular to two right lines which cut each other, it will be perpendicular to the plane of theſe right lines.



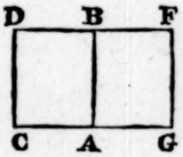
Let the line AB make right angles with the lines AC, AD; it will be perpendicular to the plane which paſſes through theſe lines.

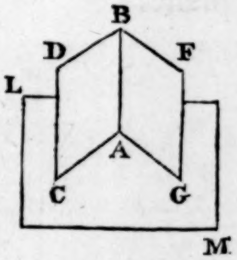
If the line AB were not perpendicular to the plane FDCG, another plane might be made to paſs through the point A, to which the line AB would be perpendicular. But this is impoſſible: for ſince the angles BAC, BAD are right angles, this other plane [def. 2.] muſt paſs through the points C, D; it would therefore 93 be the ſame with the plane FDCG, ſince theſe two planes would have three common points A, C, D.

PROP.

PROP. XCV.

From a given point in a plane, to raiſe a perpendicular to that plane.





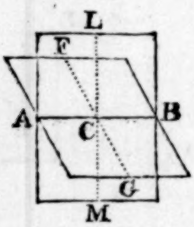
Let it be required to raiſe a perpendicular from the point A in the plane LM.

Form a rectangle CDFG, divide it into two rectangles, having a common ſection AB; and place theſe rectangles upon the plane LM in ſuch manner that the baſes of the two rectangles AC, AG ſhall be in the plane LM, and form any angle with each other; the line AB ſhall be perpendicular to the plane LM.

The line AB makes right angles with the two lines AC, AG, which by ſuppoſition are in the plane LM: it is therefore 94 perpendicular to the plane LM.

PROP. XCVI.

Two planes cutting each other at right angles, if a right line be drawn in one of the planes perpendicular to their common interſection, it will be perpendicular to the other plane.



Let the two planes AFBG, ALBM cut each other at right angles: if the line LC be perpendicular to their common interſection, it is alſo perpendicular to the plane AFBG. Draw CG perpendicular to AB.

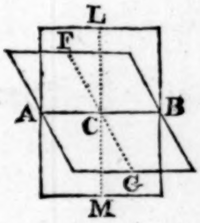
Becauſe the lines CL, CG are perpendicular to the common interſection AB, the angle LCG [def. 3.] is the angle of inclination of the two planes. Since the two planes cut each other perpendicularly, the angle of inclination LCG is therefore a right angle,

And becauſe the line LC is perpendicular to the two lines CA, CG in the plane AFBG, it is 94 perpendicular to the plane AFBG.

PROP.

PROP. XCVII.

If one plane meets another plane, it makes angles with that other plane, which are together equal to two right angles.



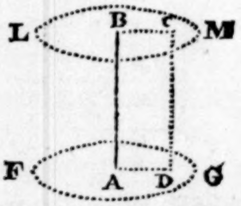
Let the plane ALBM meet the plane AFBG; theſe planes will make with each other two angles which will together be equal to two right angles. Through any point C, draw the lines FG, LM perpendicular to the line AB.

The line CL makes with the line FG two angles together equal to two right angles. But theſe two angles are [def. 3.] the angles of inclination of the two planes. Therefore the two planes make angles with each other, which are together equal to two right angles.

COR. It may be demonſtrated in the ſame manner that planes which interſect each other, have their vertical angles equal, that parallel planes have their alternate angles equal, &c.

PROP. XCVIII.

If two planes are parallel to each other, a right line which is perpendicular to one of the planes will alſo be perpendicular to the other.



Let the two planes LM, FG be parallel. If the line BA be perpendicular to the plane FG, it will alſo be perpendicular to the plane LM. From any point C in the plane LM draw CD perpendicular to the plane FG, and draw BC, AD.

Becauſe the lines BA, CD are perpendicular to the plane FG, the angles A, D are right angles.

Becauſe the planes LM, FG are parallel, the perpendiculars AB, DC [def. 6.] are equal; from whence it follows, that the lines BC, AD are parallel.

The line BA, being at right angles to the line AD, will alſo [13] be at right angles to the parallel line BC. The line BA is therefore perpendicular to the line BC.

In the ſame manner it may be demonſtrated that the line BA is at right angles to all other lines which can be drawn from the point B in the plane LM. Therefore [def. 2.] the line BA is perpendicular to the plane LM.

OF

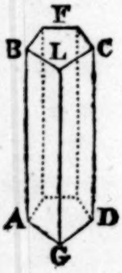
SOLIDS.

DEFINITIONS.

1. A *Solid*, as we have ſaid, is that which has length, breadth, and thickneſs.

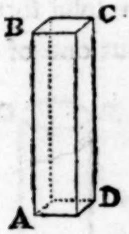
2. A *Polyhedron*, is a ſolid terminated by plane ſurfaces.

3. A *Priſm*, is a ſolid terminated by two identical plane baſes parallel to each other, and by ſurfaces which are parallelograms.

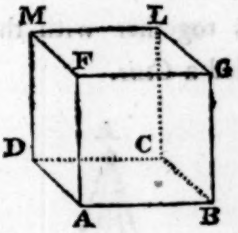


4. A *Paral-*

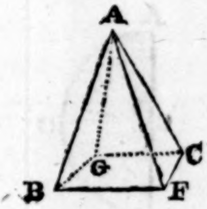
4. A *Parallelopiped*, is a priſm, the baſes of which are parallelograms.



5. A *Cube*, is a ſolid terminated by ſix ſquare ſurfaces: a die, for example, is a cube.

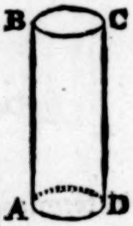


6. If right lines be raiſed from every point in the perimeter of any rectilineal figure, and meet in one common point, theſe lines together with the rectilineal figure, incloſe a ſolid which is called a *Pyramid*.

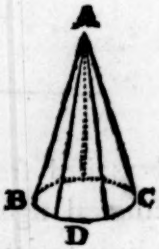


7. A *Cylinder*

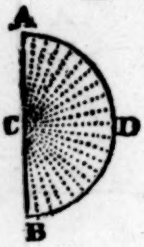
7. A *Cylinder* is a ſolid terminated by two baſes which are equal and parallel circles, and by a convex ſurface; " or, it is a ſolid formed by the revolution of a parallelogram about one of its ſides."



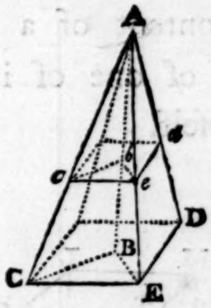
8. If right lines be raiſed from every point in the circumference of a circle, and meet in one common point, theſe lines together with the circle incloſe a ſolid which is called a *Cone.*



9. A ſemicircle, revolving about its diameter, forms a ſolid which is called a *Sphere.*



10. If from the vertex of a ſolid, a perpendicular be let fall upon the oppoſite plane, this perpendicular is called the *Altitude* of the ſolid. In the pyramids ACD, A*cd*, AB, A*b* are their reſpective altitudes.



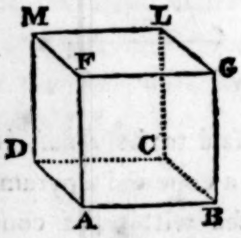
11. Solids are ſaid to be *equal*, if they incloſe an equal ſpace: thus a cone and a pyramid are equal ſolids if the ſpace incloſed within the cone be equal to the ſpace incloſed within the pyramid.

12. *Similar ſolids* are ſuch as conſiſt of an equal number of phyſical points, diſpoſed in the ſame manner.

Thus [in the fig. def. 10.] the larger pyramid ACD, and the ſmaller pyramid A*cd* are ſimilar ſolids, if every point in the larger pyramid has a point correſponding to it in the ſmaller pyramid. A hundred muſket-bullets, and a hundred cannon-balls, diſpoſed in the ſame manner, form two ſimilar ſolids.

PROP. XCIX.

The ſolid content of a cube is equal to the product of one of its ſides twice multiplied by itſelf.



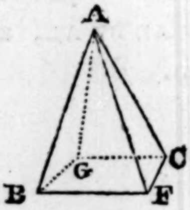
Let the lines AD, AB be equal. Let the line AD, drawn perpendicular to AB, be ſuppoſed to move through the whole length of AB: when it is arrived at BC, and coincides with it, it will have formed the ſquare DABC, and will have been multiplied by the line AB.

Next, let the line AF be drawn equal to AD and perpendicular to the plane DABC; and ſuppoſe the plane DABC to move perpendicularly through the whole length of the line AF; when it is arrived at the plane MFGL and coincides with it, it will have formed the cube AFLC, and will have been multiplied by the the line AF.

Hence it appears, that to form the cube AFLC, it is neceſſary firſt to multiply the ſide AD by the ſide AB, equal to AD; and then to multiply the product, that is the ſquare of AC, by the ſide AF equal to AD; that is, it is neceſſary to multiply AD by AD, and to multiply the product again by AD.

PROP. C.

Similar ſolids have their homologous lines proportional.



Let the two ſolids A, *a* be ſimilar; and let their homologous lines be AB, *ab*, BG, *bg*: AB will be to BG, as *ab* to *bg*.

Becauſe the ſolids A, *a* are ſimilar, every point in the ſolid A has a point correſponding to it, and diſpoſed in the ſame manner, in the ſolid *a*. Thus if the line AB is compoſed of 20 phyſical points, and the line BG of 10, the line *ab* will be compoſed of 20 correſponding points, and the line *bg* of 10. Now it is evident, that 20 is to 10 as 20 to 10: therefore AB is to BG as *ab* to *bg*.

PROP.

PROP. CI.

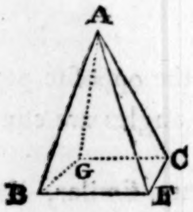
Similar ſolids are equiangular.

Let the ſolids [ſee fig. on the oppoſite page] A, *a* be ſimilar; their correſponding angles are equal.

Becauſe the ſolids A, *a*, are ſimilar, the ſurfaces BAF, *baf*, are compoſed of an equal number of points diſpoſed in the ſame manner. Theſe ſurfaces are therefore ſimilar figures, and conſequently [88] equiangular. The angles B, A, F are therefore equal to the angles *b*, *a*, *f*. In the ſame manner it may be demonſtrated that the other correſpondent angles are equal.

PROP. CII.

Solids which have their angles equal and their ſides proportional, are ſimilar.

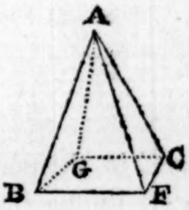


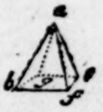
If the ſolids A, *a* have their angles equal and their ſides proportional, they are ſimilar.

For if the ſolids A, *a* were not ſimilar, another ſolid might be formed upon the line BF ſimilar to the ſolid *a*. But this is impoſſible. For, in order to form this other ſolid, ſome angle or ſome ſide of the ſolid A muſt be increaſed or diminiſhed; and then this new ſolid would not have all its angles equal, and all its ſides proportional, to thoſe of the ſolid *a:* that is, [100] [101] would not be ſimilar.

PROP. CIII.

Similar ſolids are to one another as the cubes of their homologous ſides.

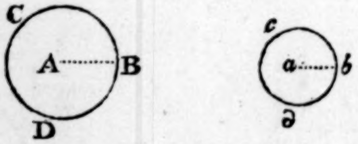




Let A, *a* be two ſimilar ſolids: the ſolid A contains the ſolid *a*, as many times as the cube formed upon the ſide BF contains the cube formed upon the ſide *bf*.

Becauſe the ſolid A is ſimilar to the ſolid *a*, every point in the ſolid A, has its correſponding point in the ſolid *a*. From whence it follows that if the ſide BF is compoſed, for example, of 50 points, the ſide *bf* will alſo be compoſed of 50 points: and conſequently, the cubes formed upon the ſides BF, *bf* will be compoſed of an equal number of points.

Let it then be ſuppoſed that the ſolid A is compoſed of 4000 points, and the cube of the ſide BF of 5000 points; the ſolid *a* muſt be compoſed of 4000 points, and the cube of the ſide *bf* of 5000 points. Now it is evident that 4000 is to 5000 as 4000 to 5000. Therefore the ſolid A is to the cube of BF, as the ſolid *a* to the cube of *bf*: and alternately, the ſolid A is to the ſolid *a*, as the cube of BF to the cube of *bf*.

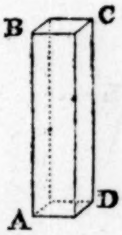


Cor. It may be demonſtrated in the ſame manner, that the ſpheres A, *a*, which are ſimilar ſolids, are to one another as the cubes of their radii AB, *ab*.

PROP.

PROP. CIV.

The ſolid content of a perpendicular priſm, is equal to the product of its baſe and height.



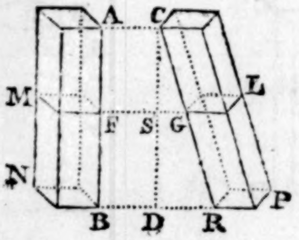
The ſolid content of the perpendicular priſm ABCD, is equal to the product of its baſe AD and height AB.

If the lower baſe AD be ſuppoſed to move perpendicularly along the height AB, till it coincides with the upper baſe BC, it will have formed the priſm ABCD. Now, the baſe AD will have been repeated as many times as there are phyſical points in the height AB. Therefore the ſolid content of the priſm ABCD, is equal to the product of the baſe multiplied by the height.

Cor. In the ſame manner it may be demonſtrated, that the ſolid content of the perpendicular cylinder ABCD, is equal to the product of its baſe AD and height AB.

PROP. CV.

The ſolid content of an inclined priſm, is equal to the product of its baſe and height.



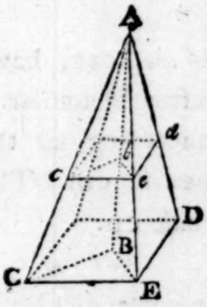
Let the inclined priſm be CP; it is equal to the product of its baſe RP and height CD.

Conceive the baſe NB of the perpendicular priſm NA, and the baſe RP of the inclined priſm PC, to move on, in the ſame time, parallel to themſelves; when they have reached the points A and C, each of them will have been taken over again the ſame number of times. But the baſe NB will have been taken over again [104] as many times as there are phyſical points in the height CD. The baſe RP will therefore have been taken over again as many times as there are phyſical points in CD. Conſequently the ſolid content of the inclined priſm CP, is equal to the product of its baſe RP and height CD.

PROP.

PROP. CVI.

In a pyramid, a ſection parallel to the baſe is ſimilar to the baſe.



Let the ſection *cd* be parallel to the baſe CD; this ſection is a figure ſimilar to the baſe. Draw AB perpendicular to the baſe CD; draw alſo BC, *bc*; BE, *be*.

Becauſe the planes *cd*, CD are parallel, AB, being perpendicular to the plane CD, will alſo [98] be perpendicular to the plane *cd*: whence the triangles A*bc*, ABC, having the angles *b*, B right angles, and the angle A common, are equiangular. Therefore [61] A*b* is to AB as *bc* to BC, and as A*c* to AC.

In like manner it may be proved that A*b* is to AB as *be* to BE, and as A*e* to AE. Conſequently, if

A*b* is one third part of AB, *bc* will be one third part of BC, *be* the ſame of BE, A*c* of AC, and A*e* of AE.

Again, in the two triangles, *c*A*e*, CAE there are, about the angle A, common to both, two ſides proportional; they are therefore 63 equiangular, and conſequently 61 have their other ſides proportional. Therefore *ce* will alſo be one third part of CE.

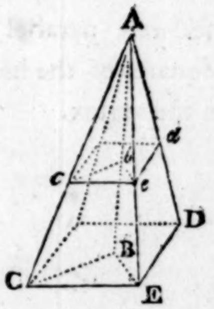
The two triangles *cbe*, CBE, having their ſides proportional, are therefore 89 ſimilar. The ſame may be demonſtrated concerning all the other triangles which form the planes *cd*, CD. Therefore the ſection *cd* is ſimilar to the baſe CD.

REMARK. If the perpendicular AB falls out of the baſe, by drawing right lines from the points *b*, B, it may be demonſtrated in the ſame manner, that the ſection is ſimilar to the baſe.

PROP.

PROP. CVII.

In a pyramid, ſections parallel to the baſe are to one another as the ſquares of their heights.



Let CD, *cd* be parallel ſections. From the vertex A draw a perpendicular AB to the plane CD: the plane *cd* is to the plane CD, as the ſquare of the height A*b* is to the ſquare of the height AB. Draw BC, *bc*.

The line AB, being perpendicular to the plane CD, will alſo [98] be perpendicular to the parallel plane *cd*: whence the angle A*bc* is a right angle, and alſo the angle ABC: moreover, the angle at A is common to the two triangles A*bc*, ABC: theſe two triangles are therefore equiangular. Therefore [61] the ſide *cb* is to the ſide CB, as the ſide A*b* to the ſide AB; and conſequently,

ſequently, the ſquare of *cb* is to the ſquare of CB, as the ſquare of A*b* to the ſquare of AB.

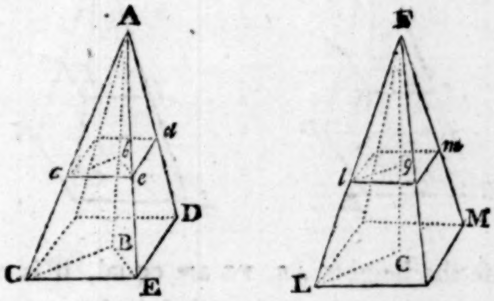
The planes *cd* CD, being [106] ſimilar figures, are to one another [82] as the ſquares of the homologous lines *cb*, CB; they are therefore alſo as the ſquares of the heights A*b*, AB.

COR. In the ſame manner it may be demonſtrated, that in a cone the ſections parallel to the baſe are to one another as the ſquares of the heights or perpendicular diſtances from the vertex.

PROP.

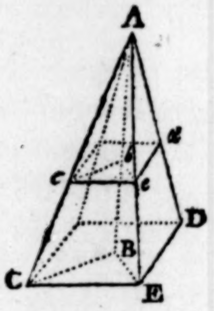
PROP. CVIII.

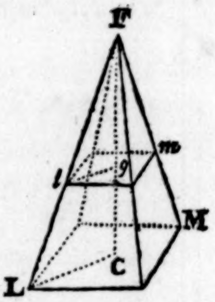
Pyramids of the ſame height are to one another as their baſes.



Let A, F, be two pyramids. If the perpendicular AB is equal to the perpendicular FG, the pyramid A is to the pyramid F, as the baſe CD to the baſe LM. Suppoſing, for example, the baſe CD to be triple of the baſe LM, the pyramid A will be triple of the pyramid F.

Two ſections, *cd*, *lm*, being taken at equal heights A*b*, F*g*; the ſection *cd* is [107] to the baſe CD, as the ſquare of the height A*b* to the ſquare of the height AB: and the ſection *lm* is to the baſe LM, as the ſquare of the height F*g* to the ſquare of the height FG. And becauſe the heights are equal, AB to FG, and A*b* to F*g*, the ſection *cd* is to the baſe CD, as the ſection *lm* to the baſe LM; and alternately, the ſection *cd* is to the ſection *lm*, as the baſe CD is to the baſe LM. But the baſe CD is triple of the baſe LM, therefore the ſection *cd* is alſo triple of the ſection *lm*.





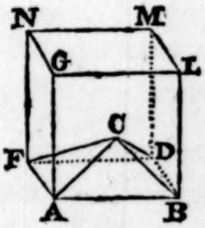
Becauſe the heights AB, FG are equal, it is manifeſt that the two pyramids are compoſed of an equal number of phyſical ſurfaces placed one upon another. Now it may be demonſtrated in the ſame manner, that every ſurface or ſection of the pyramid A is triple of the correſponding ſurface or ſection of the pyramid F. Therefore the whole pyramid A is triple of the whole pyramid F.

COR. Hence it follows, that pyramids of the ſame height and equal baſes, are equal; ſince they are to one another as their baſes.

PROP.

PROP. CIX.

A pyramid, whoſe baſe is that of a cube, and whoſe vertex is at the center of the cube, is equal to a third part of the product of its height and baſe.



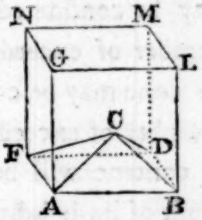
Let the cube AM and the pyramid C have the ſame baſe AD, and let the vertex of the pyramid be at the center of the cube C; this pyramid is equal to a third part of the product of its height and baſe.

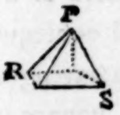
Conceive right lines drawn from the center of the cube to its eight angles A, B, D, F, N, G, L, M; the cube will be divided into ſix equal pyramids, each of which has one ſurface of the cube for its baſe, and half the height of the cube for its height; for example, the pyramid CABDF.

Three of theſe pyramids will therefore be equal to half the cube. Now the ſolid content of half the cube is equal to the product of the baſe and half the height: [compare Prop. 99.] Each pyramid will therefore be equal to one third part of the product of the baſe and half the height of the cube, that is the whole height of the pyramid.

PROP. CX.

The ſolid content of a pyramid is equal to a third part of the product of its height and baſe.





Let RPS be a pyramid; its ſolid content is equal to a third part of the product of its height and its baſe RS.

Form a cube, the height of which, BL, is double of the height of the pyramid RPS. A pyramid, the baſe of which is that of this cube, and the vertex of which is C the center of the cube, will be equal to a third part of the product of its baſe and height.

The pyramids C and P have the ſame height; they are therefore [Prop. 108. Cor.] to one another as their baſes. If the baſe AFDB is double of the baſe RS, the pyramid C will therefore be double of the pyramid P.

But the pyramid C is equal to a third part of the product of its height and baſe. The pyramid P will therefore be equal to a third part of the product of the ſame height and half the baſe AFDB, or, which is the ſame thing, the whole baſe RS.

PROP.

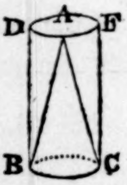
PROP. CXI.

The ſolid content of a cone is equal to a third part of the product of its height and baſe.

For the baſe of a cone may be conſidered as a polygon compoſed of a great number of exceedingly ſmall ſides: and conſequently the cone may be conſidered as a pyramid having a great number of exceedingly ſmall ſurfaces: whence its ſolid content will be equal 110 to one third part of the product of its height and baſe.

PROP. CXII.

The ſolid content of a cone is a third part of the ſolid content of a cylinder deſcribed about it.

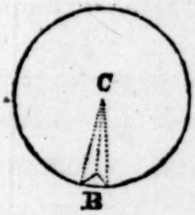


Let the cone BAC, and the cylinder BDFC have the ſame height and baſe, the cone is a third part of the cylinder.

For the cylinder is equal to the product of its height and baſe; and the cone is equal to a third part of this product: therefore the cone is a third part of the cylinder.

PROP CXIII.

The ſolid content of a ſphere is equal to a third part of the product of its radius and ſurface.



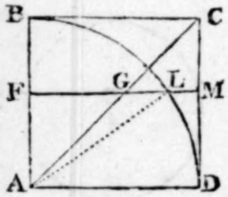
Two points not being ſufficient to make a curve line, three points will not be ſufficient to make a curve ſurface. If therefore all the phyſical points which compoſe the ſurface of the ſphere C be taken, three by three, the whole ſurface will be divided into exceedingly ſmall plane ſurfaces: and radii being drawn to each of theſe points the ſphere will be divided into ſmall pyramids which have their vertex at the center, and have plane baſes.

The ſolid contents of all theſe ſmall pyramids will be equal 110 to a third part of the product of the height and baſes. Therefore the ſolid content of the whole ſphere will be equal to a third part of the product of the height and all the baſes, that is, of its radius and ſurface.

PROP.

PROP. CXIV.

The ſurface of a ſphere is equal to four of its great circles.



If a plane biſect a ſphere, the ſection will paſs through the center, and is called a great circle of the ſphere.

Let ABCD be a ſquare; deſcribe the fourth part of the circumference of a circle BLD; draw the diagonal AC, the right line FM parallel to AD, and the right line AL.

In the triangle ABC, on account of the equal ſides AB, BC, the angles A and C are [4] equal; therefore ſince the angle B is a right angle, the angles A and C are each half a right angle. Again in the triangle AFG, becauſe the angle F is a right angle, and the angle A half a right angle, the angle G is alſo half a right angle: therefore [26] AF is equal to FG.

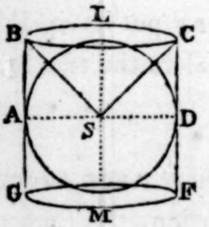
The radius AL is equal to the radius AD; but AD is equal to FM; therefore AL is equal to FM.

In the rectangular triangle AFL, the ſquare of the hypothenuſe AL is equal 32 to the two ſquares of AF and FL taken together. Inſtead of AL put its equal FM, and inſtead of AF put its equal FG; and the ſquare of FM will be equal to the two ſquares of FG and FL taken together.

Conceive the ſquare ABCD to revolve about the line AB. In the revolution, the ſquare will deſcribe a cylinder, the quadrant a hemiſphere, and the triangle ABC an inverted cone, the vertex of which will be in A. Alſo, the line FM will form a circular ſection of a cylinder, the line FL will form a circular ſection of a hemiſphere, and the line FG a circular ſection of a cone.

Theſe circular ſections, or circles are to each other 81 as the ſquares of their radii: therefore, ſince the ſquare of the radius FM is equal to the ſquares of the two radii FL and FG, the circular ſection of the cylinder will be equal to the circular ſections of the hemiſphere and cone.

In the ſame manner it may be demonſtrated, that all the other ſections, or circular ſurfaces of which the cylinder is compoſed, are equal to the correſponding ſections, or ſurfaces, of the hemiſphere and cone. Therefore the cylinder is equal to the hemiſphere and cone taken together: but the cone [112] is equal to a third part of the cylinder; the hemiſphere therefore is equal to the remaining two thirds of the cylinder; and conſequently the hemiſphere is double of the cone.



The cone BSC is [111] equal to a third part of the product of the radius and the baſe BC, which is a great circle of the ſphere: the hemiſphere ALD is therefore equal to a third part of the product of the radius and two of its great circles; and conſequently, the whole ſphere is equal to a third part of the product of the radius and four of its great circles.

Laſtly, ſince the ſphere is equal [113] to a third part of the product of the radius and ſurface of the ſphere, and alſo to a third part of the product of the radius and four of its great circles, the ſurface of the ſphere is equal to four of its great circles.

THE END.

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